Singular points in planar dynamical systems and in FLRW metric $f(R)$ cosmology models

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General Relativity

General Relativity (GR) and Einstein's field equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \kappa T_{ab} \tag{1}$$

provide the current more reliable description of gravitation in Physics.

General Relativity

Einstein's field equations can be derived by varying the action with cosmological constant

$$S = S_G(g_{ab}) + S_M(g_{ab}, \psi) = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(-2\Lambda + R) + S_M(g_{ab}, \psi), \qquad (2)$$

where

$$R := g^{ab}R_{ab} \qquad (3)$$

is the *curvature scalar* and the *stress–energy–momentum* tensor of matter fields is

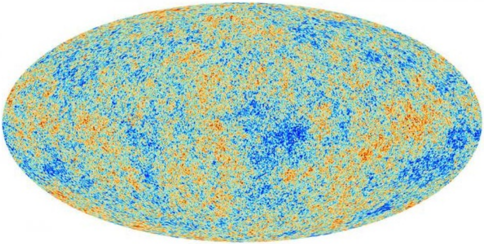
$$T_{ab} := \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}}. \qquad (4)$$

Modified Theories of Gravity

Despite the numerous successes of General Relativity (GR), there are indications that the theory is "incomplete"; the main one is

- The incompatibility of GR with Quantum Mechanics.

After Utiyama and de Witt showed in 1962 that renormalization requires higher–order curvature terms in $I_G(g_{ab})$, the *Higher-order theories of Gravity* became popular.



Metric $f(R)$ theories of gravity

One of the most common choices is to investigate a more general dependence on R within the Lagrangian of gravitational actions:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{ab}, \psi). \tag{5}$$

After the variation with respect to g^{ab}, and some manipulations one finds the fields equations:

$$f_R(R) R_{ab} - \frac{1}{2} f(R) g_{ab} - (\nabla_a \nabla_b - g_{ab} \Box) f_R(R) = \kappa T_{ab}, \tag{6}$$

where $\Box := \nabla^a \nabla_a$, and ∇_a denotes the covariant derivative with respect to the Levi–Civita connection.

Friedmann–Lemaitre–Robertson–Walker (FLRW) spacetime

As in the usual GR scenario, one can investigate homogeneous and isotropic solutions of (6), by considering the metric ansatz

$$g_{ab}\,dx^a dx^b = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{7}$$

where (r,θ,ϕ) are comoving coordinates, and $a(t)$ is a scale factor of the universe. After introducing the Hubble parameter, one is left with

$$\dot{R} = \frac{1}{3f_{RR}(R)H}\left[\kappa\rho + \frac{f_R(R)R - f(R)}{2} - 3f_R(R)H^2 - f_R(R)\frac{K}{a^2}\right], \tag{8}$$

$$\dot{H} = \frac{R}{6} - 2H^2 - \frac{K}{a^2}, \tag{9}$$

$$\dot{\rho} = -3H(\rho + p), \tag{10}$$

$$\dot{a} = Ha.$$

If we consider a spatially flat universe in the absence of matter, the latter system decouples, and one reduces to

$$\begin{cases} \dot{R} &= \dfrac{1}{f_{RR}(R)H}\left[\dfrac{1}{6}\Big(f_R(R)R - f(R)\Big) - f_R(R)H^2\right], \\ \dot{H} &= R/6 - 2H^2, \end{cases} \tag{12}$$

where $f \in C^3(I)$, and $f_{RR} > 0$ in the open interval $I \subseteq \mathbb{R}$.

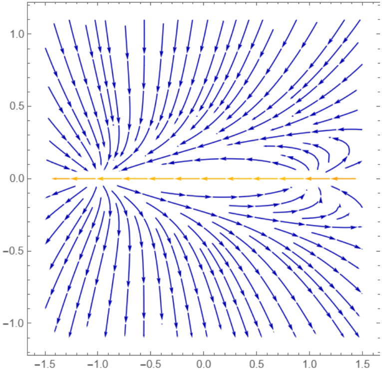


Figure: Phase space in the (R, H)–coordinates of the above equations where $f(R) = 1 + R + R^2$.

Singular autonomous planar dynamical systems

Numerous contemporary models in the fields of physics, demographics, and engineering revolve around the use of singular systems of ordinary differential equations (ODEs). In this presentation, we address the autonomous planar case:

$$\begin{aligned} \alpha(x,y)\,\dot{x} &= \beta(x,y) \\ \dot{y} &= \delta(x,y), \end{aligned} \tag{13}$$

where $\alpha, \beta, \delta \in C^1(\Omega)$, mainly aiming at describing the qualitative dynamical features of Carathéodory solutions near the **singular curve**

$$\mathtt{SC} := \{(x,y) \in \Omega \mid \alpha(x,y) = 0\}. \tag{14}$$

Singular points in the Geometric Theory of Differential Equations

The definition that best formalizes our intuition of singular point is given in the context of the Geometric Theory of Implicit Differential Equations, i.e. expressions of the form

$$G(x, y, p) = 0, \tag{15}$$

where $p = dy/dx$ and G is a smooth function in some domain of $\mathbb{R}^3$.

Definition

A point of the surface $\mathscr{M}$, defined by (15), is said to be *singular*, if it is a critical point of the vertical projection mapping onto the (x, y)–plane.

By the implicit function theorem, the singular points form a curve, the *criminant curve*, in the three–dimensional space of 1–jets:

$$\mathtt{CC} = \{(x, y, p) \mid G(x, y, p) = 0, \quad G_p(x, y, p) = 0\}. \tag{16}$$

In our case, we can view the solutions of (13) as a particular parametrization of the integral curves of the implicit differential equation

$$G(x,y,p) = \beta(x,y)p - \alpha(x,y)\delta(x,y) = 0, \tag{17}$$

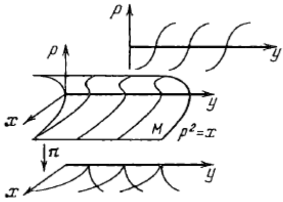
as

$$p = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\alpha(x,y)\delta(x,y)}{\beta(x,y)}. \tag{18}$$

It follows that the criminant curve of (17) is

$$\mathtt{CC} = \{(x,y,p) \mid \beta(x,y) = 0, \quad \alpha(x,y)\delta(x,y) = 0\}, \tag{19}$$

which, in the generic case, is composed of straight lines parallel to the p–axis whose projections on the (x,y)–plane are equilibrium points or lie on $\mathtt{SC}$.



Regular associated system

- The main idea is to deduce the qualitative properties of Carathéodory solutions of (13), by the **regular associated dynamical system** defined as

$$\begin{cases} \dot{x} = \beta(x, y), \\ \dot{y} = \delta(x, y)\alpha(x, y), \end{cases} \tag{20}$$

  as the solutions of the two systems are two different parametrizations of the same integral curves.

- In fact, they have the same direction $p = dy/dx$ in the phase space, and their "time arrows" are reversed only in

$$\Omega_- := \{(x, y) \in \Omega \mid \alpha(x, y) < 0\}. \tag{21}$$

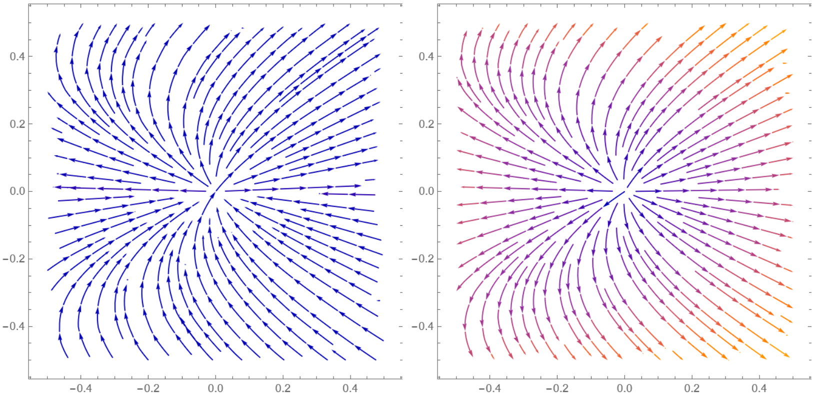


Figure: The phase portrait of the singular system $\dot{x} = \frac{x-2(x-1)y^2}{y}$, $\dot{y} = 1$ and the one of the regular associated system $\dot{x} = x - 2(x-1)y^2$, $\dot{y} = y$.

Singular points

- We want to study the behaviour of solutions in a neighborhood of singular points, i.e. points where the integral curves are not (locally) parallel to each other.
- The introduction of the regular associated system, suggests an intuitive definition:

Definition

- A point $\boldsymbol{x}_s \in \Omega$ is a **proper singular point** of (13), if it is an equilibrium point of the regular associated system (20), but not of (13).
- A **nondegenerate (proper) singular point** of (13) is a singular point $\boldsymbol{x}_s \in \Omega$ such that

$$\det[D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)] \neq 0. \tag{22}$$

Non–transversality condition

Motivated by our starting example, from now on, we require the *non–transversality condition*

$$\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \equiv 0 \quad \text{in} \quad \mathtt{SC}, \tag{23}$$

as this implies

$$\langle \tilde{\boldsymbol{F}}(\boldsymbol{x}), \boldsymbol{\nabla} \alpha(\boldsymbol{x}) \rangle = \beta(\boldsymbol{x}) \frac{\partial \alpha}{\partial x}(\boldsymbol{x}) = 0, \tag{24}$$

i.e., up to a coordinate change and restricting the domain, the singular curve is the x–axis in Ω.

Proposition

Assume $\alpha = 0$ and $\beta = 0$ have a discrete intersection, and they do not cross $\delta = 0$ simultaneously. If the non–transversality condition holds, a Carathéodory solution of (13) *can evolve through the singular curve $\alpha = 0$, iff it passes through a singular point.*

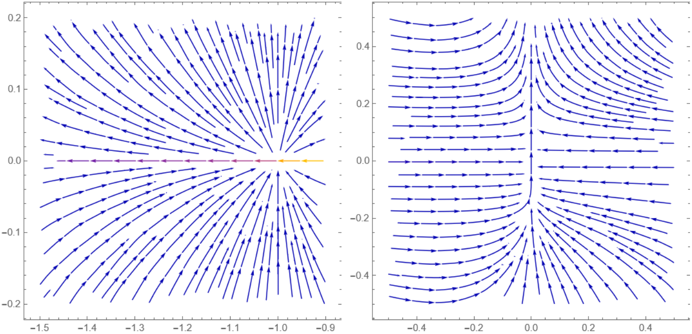


Figure: On the left: the phase portrait of $\dot{x} = (1+x)/y$, $\dot{y} = x^2$. On the right: the one of the system $\dot{x} = -x/y^2$, $\dot{y} = 1 + 3x$.

Definition

A point $\bar{x}$ on the singular curve is called a **funnel point** if there is an open neighborhood V of $\bar{x}$ such that

1) no Carathéodory solution of (13) with initial data in V can cross the singular curve if not passing through $\bar{x}$.

2i) all the Carathéodory solutions of (13) originating in

$$V_+ := V \cap \Omega_+ \tag{25}$$

contain such a point in the future [resp. past];

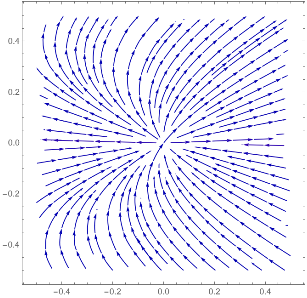
2ii) all the Carathéodory solutions of (13) originating in

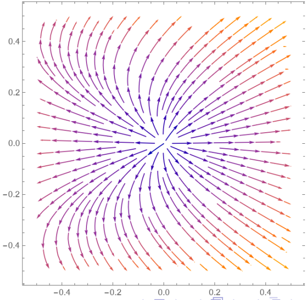
$$V_- := V \cap \Omega_- \tag{26}$$

contain such a point in the past [resp. future].

Theorem

Assume $\alpha = 0$ and $\beta = 0$ have a discrete intersection, and they do not cross $\delta = 0$ simultaneously. Suppose that the singular system (13) has a singular point $\boldsymbol{x}_s$ on $\mathcal{SC}$ and that $z = \alpha(x, y)$ properly intersects the (x, y)–plane on the x–line, i.e. the non–transversality condition (23) holds in Ω. If $\boldsymbol{x}_s$ corresponds either to a (locally) asymptotically stable/unstable node of the regular system (20), then the singular point $\boldsymbol{x}_s$ is a funnel point.





Remark

- For simplicity the above theorem is stated with respect to the reduced form (13), notice however that in a more general case, when the singular curve coincides with the separatrix of the node, one can reduce to (13) by a coordinate change.
- By the principle of linearized stability follows that the last condition, in the above statement, holds if all the eigenvalues of $D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)$ have negative (or positive) real parts.

Solution continuation

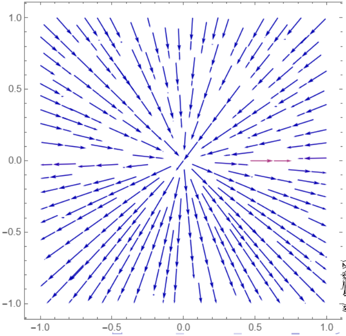
- It is interesting to consider additional conditions at the singular points so as not to lose the uniqueness of solutions.
- A natural condition that allows to define C^1 solutions, is the one of prescribing to "keep the same direction in the phase plane".

If we consider

$$\begin{cases} \dot{x} = -x/y \\ \dot{y} = -1. \end{cases} \tag{27}$$

the quantity x/y is a constant of motion as

$$\ddot{x} = -\frac{-\frac{x}{y} \cdot y + x \cdot 1}{y^2} \equiv 0. \tag{28}$$



Case 1: Incident flow

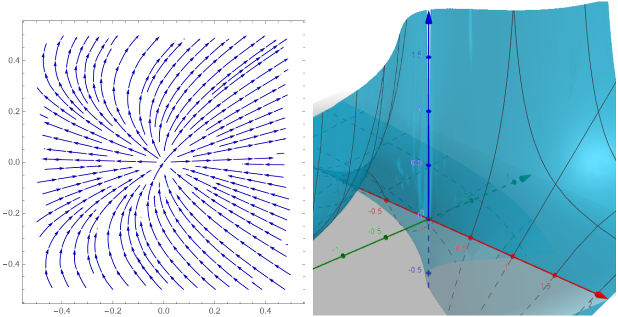


Figure: One the left, the phase portrait of the singular system $\dot{x} = \frac{x-2(x-1)y^2}{y}$, $\dot{y} = 1$ around the origin, which is a nondegenerate singular point. In this case the eigenvalues of $D\tilde{\boldsymbol{F}}(0)$ are $\lambda_1 = \lambda_2 = 1$. On the right: the correspondent phase surface obtained lifting the phase plane in the (x, y, p)–space of 1–jets.

Case 2: Tangential flow

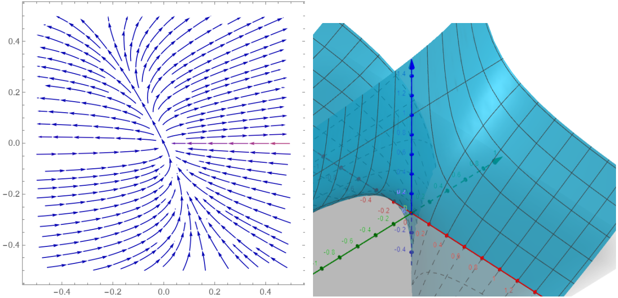


Figure: One the left, the phase portrait of the singular system $\dot{x} = \frac{3x-(x-1)y}{y}$, $\dot{y} = 1 + x$ around the origin, which is a nondegenerate singular point. In this case the eigenvalues of $D\tilde{\boldsymbol{F}}(0)$ are $\lambda_1 = (3-\sqrt{5})/2$ and $\lambda_2 = (3+\sqrt{5})/2$. On the right: the correspondent phase surface obtained lifting the phase plane in the (x, y, p)–space of 1–jets.

Singular points

Now, let us apply the previous results to our cosmological model:

$$\begin{cases} \dot{R} &= \dfrac{1}{f_{RR}(R)H}\left[\dfrac{1}{6}\Big(f_R(R)R - f(R)\Big) - f_R(R)H^2\right], \\ \dot{H} &= R/6 - 2H^2. \end{cases} \tag{29}$$

Lemma

If $f \in C^3(I)$, and $f_{RR} > 0$ in $I \subseteq \mathbb{R}$, then the equation

$$f_R(R)R = f(R) \tag{30}$$

admits at most two distinct solutions in I. Moreover, if two solutions exist they have opposite signs.

As singular points of the system (12) satisfy $H = 0$ and (30), if $f(0) \neq 0$, we conclude that there exists at most one singular point in the left/right semi–plane.

Singular points are funnels

In order to investigate the nature of singular points, we study the derivative of the regular associated system:

$$D\tilde{\boldsymbol{F}}(R,H) = \begin{pmatrix} f_{RR}(R)(R-6H^2) & -12H\cdot f_R(R) \\ H\cdot(f_{RRR}(R)(R-12H^2)+f_{RR}(R)) & f_{RR}(R)(R-36H^2) \end{pmatrix}. \qquad (31)$$

which in the singular points $(R_s, 0)$ reads $D\tilde{\boldsymbol{F}}(R_s,0) = f_{RR}(R_s)R_s\,\mathbb{I}_{2\times 2}$.

- From the Theorem 1 above follows that the singular points are funnels.

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Singular points are funnels

In conclusion, if $f \in C^3(I)$, $f(0) \neq 0$ and $f_{RR} > 0$ in the open interval $I \subseteq \mathbb{R}$, the singular points are funnel points of the first kind.

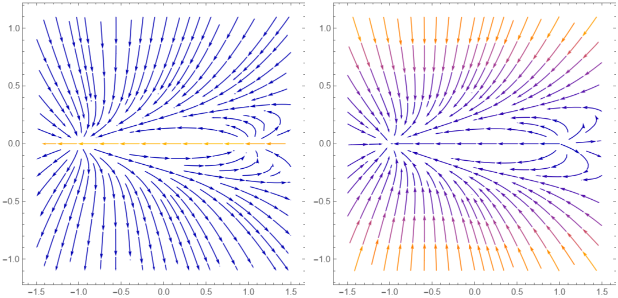
- The additional condition of preserving the direction of a solution at a singular point prescribes a unique continuation.
- This additional condition allows defining a dynamical system from the equations (12).

In the case $f(R) = a + bR + cR^2$ with $c > 0$, the singular points satisfy

$$R_s = \pm\sqrt{\frac{a}{c}}, \quad \text{and} \quad H_s = 0, \tag{32}$$

and so they exist if and only if $a > 0$. The Jacobian matrix of the regular associated system is

$$D\tilde{\boldsymbol{F}}(\pm\sqrt{a/c}, 0) = \pm 2\sqrt{ac}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$



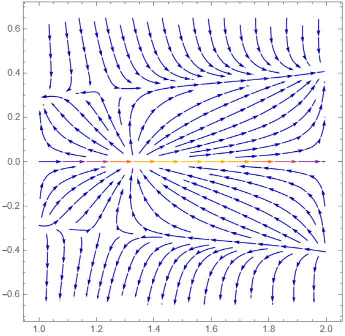
Bounded curvature models

- Since singular points may correspond to equilibria of the regular associated system possessing a basin of attraction, their importance, from a dynamical perspective, is not only local.

For instance, consider the hemi–circular bounded–curvature model

$$f(R) := -\sqrt{(R - R_r)(R_l - R)}, \qquad (34)$$

with $0 < R_r < R_l$ defined in the interval $I = (R_r, R_l)$.



Conclusion

In this work, starting from an FLRW $f(R)$ cosmology model

- we investigated the nature of singular points of planar autonomous systems of ODEs of the form (13) by studying the regular associated system (20).
- We defined *funnel points* and showed sufficient conditions for a singular point to be a funnel.
- Additional conditions at the singularity were discussed in relation to the Geometric Theory of differential equations.
- The above discussion was applied to the initial example leading to the conclusion that singular points of (12) are funnel points (under the usual assumptions on f and when $f(0) \neq 0$).

Possible Future Directions

- Investigate symmetric periodic orbits of the FLRW model (12).
- Examine the particular dynamics by selecting specific and appropriate forms of $f(R)$.
- Implementing the aforementioned methods on different systems, even in higher dimensions.

Thanks for your attention!