Algebraic Curves and Riemann–Roch Theorem

Presented by:

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Overview

In this presentation we expose the ideas behind the **algebraic** definition of the concept of **genus** for an algebraic curve over $\mathbb{K}$. In particular

- if C is smooth and projective over $\mathbb{C}$, we want that the latter coincides with the topological genus of $C(\mathbb{C})$.
- We see that it will emerge algebraically by treating the group of **Divisors** of the curve.
- Finally we will concentrate on Riemann–Roch Theorem.

Plane Curves and Singularities

Algebraic plane curves correspond to the zero set of a nonconstant $C \in \mathbb{K}[x, y]$.

- If C is irreducible, $V(C)$ is an irreducible variety in $\mathbb{A}^2_{\mathbb{K}}$ so we restrict to this case.
- A point $P = (a, b) \in C$ **smooth** or **simple** if $\partial_x C(P) \neq 0$ or $\partial_y C(P) \neq 0$, in this case its tangent line is

$$\partial_y C(P)(x - a) + \partial_y C(y - b) = 0.$$

- Otherwise P is called a **singular** point, in particular: it is an **ordinary singularity** if its tangent cone is composed of distinct lines.
- The **multiplicity** $m_P(C)$ of an ordinary singularity P is the number of lines in its tangent cone.

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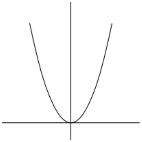
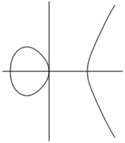
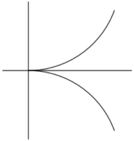


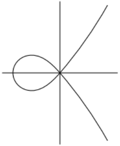
$A = Y - X^2$



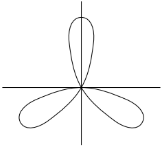
$B = Y^2 - X^3 + X$



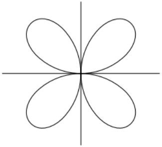
$C = Y^2 - X^3$



$D = Y^2 - X^3 - X^2$



$E = (X^2 + Y^2)^2 + 3X^2Y - Y^3$



$F = (X^2 + Y^2)^3 - 4X^2Y^2$

- Notice that if $P = (0, 0)$ the multiplicity of the curve in P is the degree of its the lowest homogeneus component.
- After a change of coordinate one can use the same criterion to find the multiplicity of the other singular points.

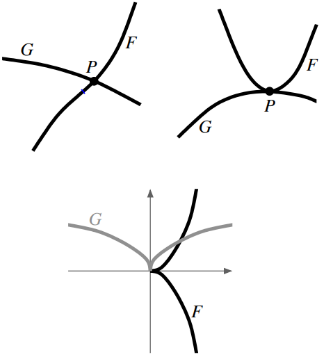
Intersection numbers

For all plane curves F and G and all points $P \in \mathbb{A}^2_{\mathbb{K}}$ there is a unique nonnegative integer

$$I_P(F,G) := \dim_{\mathbb{K}} \mathcal{O}_P(\mathbb{A}^2)/(F,G)$$

such that

- if F and G do not intersect $I_P(F,G) = 0$,
- if F and G do not intersect properly $I_P(F,G) = \infty$,
- $I_P(F,G) \geq m_P(F)m_P(G)$,
- $I_P(F,G) = I_P(F, G + AF)$ for any $A \in \mathbb{K}[x,y]$.



More generally, algebraic curves are 1–dimensional varieties, so its field of rational function has transcendence degree one.

Theorem

Any algebraic curve is birational to a plane projective curve with only ordinary singularities.

Since any algebraic curve has a smooth model, in other words:

Theorem

Any algebraic curve curve is birational to a unique (up to isomorphism) smooth projective curve.

we concentrate on smooth projective curves.

Affine curves

We consider a smooth, irreducible, affine curve C over an algebraically closed field $\mathbb{K}$, and denote

- $A(C) := \mathbb{K}[x,y]/(C)$ its **affine coordinate ring**.
- $\mathbb{K}(C) := \{f/g : f, g \in A(C) \text{ and } g \neq 0\}$ its field of **rational functions**.
- $\mathcal{O}_P(C)$ the **local ring of** P on $V(C)$, with the (unique) maximal ideal

$$\mathfrak{m}_P(C) = \{f/g : f, g \in A(C), \text{ with } f(P) = 0,\ g(P) \neq 0\}.$$

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Discrete valuation rings $\mathcal{O}_P$

Recall that, since every point of C is regular, any $\mathcal{O}_P$ is a discrete valuation ring:

a) $\mathfrak{m}_P(C) = (t)$ is principal, with $t \in \mathcal{O}_P$ a **local coordinate** for C in P.

b) Every $\varphi \in \mathbb{K}(C)$ can be written $\varphi = ct^n$, with c a unit, i.e. $c \in \mathcal{O}_P - \mathfrak{m}_P$, and $n = \text{ord}_P(\varphi)$.

c) If $\varphi = f/g$ we have $\text{ord}_P(\varphi) = I_P(f, C) - I_P(g, C)$.

d) In particular, $\text{ord}_P : \mathbb{K}(C)^* \to \mathbb{Z}$ is a discrete valuation, and $A(C)$ and $\mathfrak{m}_P$ are respectively the pullbacks of $[0, \infty)$ and $(0, \infty)$.

e) Intuitively, $\text{ord}_P(\varphi)$ is grater (resp. is less) than zero if and only if φ has a zero (resp. a pole) in P.

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Projective curves

Similarly we introduce above definitions in case of projective curves, considering $S(C)$ the **homogeneus coordinate ring** of C as a graded ring, i.e.

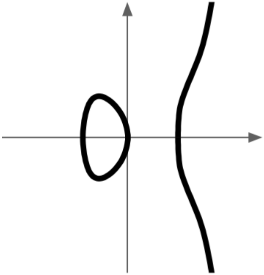
$$S(C) = \bigoplus_{d \geq 0} S_d(C).$$

The definition of multiplicity of rational functions on C is easily extended since it is a local property.

Example

Consider the rational function $\varphi = \frac{y}{x}$ over the projective curve $C : Y^2Z - X^3 + XZ^2$, and $P = (0:0:1) \in C$. Then

$$\mathrm{ord}_P(\varphi) = \mathrm{ord}_P(y) - \mathrm{ord}_P(x) = 1 - 2 = -1.$$



$y^2 + x - x^3 = 0$

Theorem (Bézout)

Let F and G be two projective curves without common component over a ground field $\mathbb{K}$, then the number of their intersections counted with multiplicity is

$$\sum_{P \in F \cap G} I_P(F, G) \leq \deg F \cdot \deg G.$$

Moreover, equality holds if $\mathbb{K}$ is algebraically closed.

Theorem (Noether)

Let F be a smooth projective curve over $\mathbb{K}$ an algebraically closed field. Let G and H be two smooth projective curves that do not have a common component with F. Then

- *if $I_P(F,G) \leq I_P(F,H)$ for all $P \in \mathbb{P}_2$ there are homogeneus polynomials A and B (of degrees $\deg H - \deg F$ and $\deg H - \deg G$) such that*

a) $H = AF + BG$;

b) $I_P(F,H) = I_P(F,G) + I_P(F,B)$ *for all* $P \in \mathbb{P}_2$.

Topology of complex Curves

If we consider a smooth projective curve over $\mathbb{C}$, its points $V(C)$ form a 1–dimensional complex manifold, moreover

- $V(C)$ is *compact*,
- $V(C)$ is *oriented*,
- $V(C)$ is *connected*.

Therefore, it homeomorphic to a sphere with a finite amount of "handles", i.e. the **topological genus** of $V(C)$.

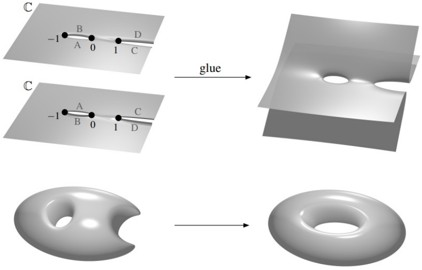


Figure: $C : ZY^2 + Z^2X - X^3 = 0.$

Topological genus of complex Curves

Cell decomposition

Definition

Let $\mathcal{M}$ be a compact 2–dimensional (real) manifold, a **cell decomposition** of $\mathcal{M}$, is a finite disjoint union of *points*, *(open) lines*, and *(open) discs*.

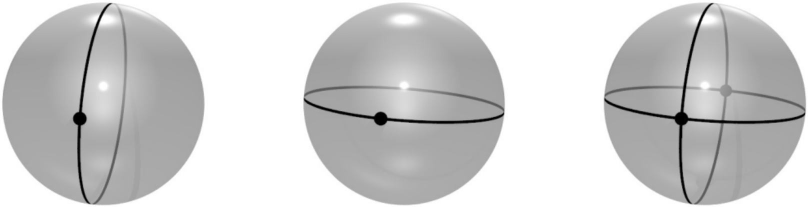


Figure: Three valid disjoint decompositions of $\mathbb{P}_1(\mathbb{C})$.

Topological genus of complex Curves

Euler characteristic

Lemma

Let $\mathcal{M}$ be a compact 2–dimensional (real) manifold. Consider a cell decomposition of $\mathcal{M}$ consisting of σ_0 points, σ_1 lines, and σ_2 discs. The number

$$\chi := \sigma_0 - \sigma_1 + \sigma_2$$

only depends on $\mathcal{M}$ and is called **Euler characteristic** of $\mathcal{M}$.

Euler characteristic

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- From any two cell decomposition we can find a common refinement.
- Such a refinement is obtained through operations of two types:
  1) adding another point on a line,
  2) adding another line to a disc,

  which do not modify χ.

Relation between g and χ

Given $\mathcal{M}$ a connected and compact oriented 2–dimensional (real) manifold, we can build a valid cell decomposition using 4 lines and 2 points for each hole which lead to a total of

- $\sigma_0 = 2g + 2$ points,
- $\sigma_1 = 4g + 4$ lines,
- $\sigma_1 = 4$ discs.

Therefore

$$\chi = \sigma_0 - \sigma_1 + \sigma_2 = 2 - 2g \quad \iff \quad g = 1 - \chi/2.$$

Theorem (Topological degree–genus formula)

A smooth curve of degree d in $\mathbb{P}_2(\mathbb{C})$ has topological genus $\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$.

Proof sketch:

- Wlog $(0:1:0) \notin C$ and the projection $\pi : V(C) \to \mathbb{P}_1(\mathbb{C})$ is well defined.
- The inverse images of the points $(x:z) \in \mathbb{P}_1(\mathbb{C})$ contains d points unless C has a multiple zero, i.e. C and $\partial_y C$ are both null there.
- Their corresponding inverse image has $d-1$ points.
- Since at a ramification point we have $I_P(C, \partial_y C) = 1$ we have a total of $\deg C \cdot \deg \partial_y C = d(d-1)$ *ramification points* of π.
- Taking a fine cell decomposition of $\mathbb{P}_1(\mathbb{C})$ containing their π-images and s.t. $\sigma_0 - \sigma_1 + \sigma_2 = 2$ and lifting it leads

$$\chi = d\sigma_0 - d(d-1) - d\sigma_1 + d\sigma_2 = 3d - d^2.$$

Topological genus of complex Curves

Summarizing

- We have seen that to every curve in $\mathbb{P}_2(\mathbb{C})$ can be assigned such a topological invariant, its genus.
- Remarkably, the topological genus of a smooth complex projective curve depends only on its algebraic degree.
- A natural question arises: "is it possible to define an **algebraic genus** for smooth projective curves which coincides with the topological one in case $\mathbb{K} = \mathbb{C}$?"

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Divisors on Curves

Let C be a smooth projective curve

- A **divisor** on C is a formal sum $D := \sum_{P \in C} a_P P$, where $a_P \in \mathbb{Z}$ and $a_P = 0$ for all but a finite amount of P.
- A divisor $D \in \text{Div } C$ is called **effective** if $a_P \geq 0$ for all $P \in C$.
- For $D_1, D_2 \in \text{Div } C$ we write

$$D_1 \geq D_2 \quad \Longleftrightarrow \quad D_1 - D_2 \text{ is effective.}$$

- The **degree** of a divisor $D := \sum_{P \in C} a_P P$ is $\deg D := \sum_{P \in C} a_P$ and we denote

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Rational functions and divisors

Having defined the multiplicities of polynomials and rational functions on curves allows to introduce a particular class of divisors.

- For a non–zero polynomial $f \in S(C) - \{0\}$, the **divisor of** f is

$$\text{div } f := \sum_{P \in C} \text{ord}_P(f) \cdot P \qquad \geq 0$$

- Similarly for a non–zero rational function $\varphi \in \mathbb{K}(C)^*$, the **divisor of** φ is

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Rational functions and divisors

Since ord_P is a valuation we have that

$$\text{div}(fg) = \sum_{P \in C} \text{ord}_P(fg) \cdot P = \sum_{P \in C} (\text{ord}_P(f) + \text{ord}_P(g)) \cdot P = \text{div } f + \text{div } g,$$

and similarly

$$\text{div}(\varphi\psi) = \text{div}(\psi) + \text{div}(\psi).$$

In particular

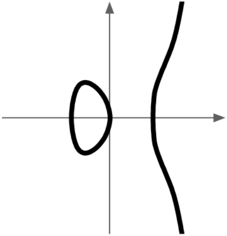
$$\text{div} : \mathbb{K}^*(C) \to \text{Div } C$$

is a group homomorphism.

Example

Consider the rational function $\varphi = \frac{y}{x}$ over the projective curve $C : Y^2Z - X^3 + XZ^2$, then

$$\begin{aligned}\text{div } \varphi &= (0:0:1) + (-1:0:1) + (1:0:1) - 2(0:0:1) - (0:1:0)\\ &= -(0:0:1) + (-1:0:1) + (1:0:1) - (0:1:0)\end{aligned}$$



$y^2 + x - x^3 = 0$

Bézout and Nother theorems for divisors

Notice that for f and φ respectively a non–zero polynomial and rational function we have

$$\deg(\text{div } f) = \sum_{P \in C} \text{ord}_P(f) = \sum_{P \in C} I_P(C, f) = \deg C \cdot \deg f,$$

while writing $\varphi = h/g$, with $h, g \in S_d(C)$ and $g \neq 0$ we get

$$\deg(\text{div } \varphi) = \deg(\text{div } h) - \deg(\text{div } g) = \deg C \cdot (d - d) = 0.$$

In particular

$$\text{div} : \mathbb{K}^*(C) \to \text{Div}^0 C \subset \text{Div } C.$$

Bézout and Nother theorems for divisors

Proposition (Noether's Theorem for divisors)

If C is a projective curve and $g, h \in S(C)$ non–zero homogeneous polynomials with div $g \leq$ div h. Then, there exists a homogeneous $b \in S(F)$ of degree $\deg h - \deg g$ such that

$$h = bg \text{ in } S(F), \quad \text{and} \quad \text{div } h = \text{div } b + \text{div } g.$$

Corollary

The only rational functions everywhere regular on a projective curve C are constants, i.e.

$$\bigcap_{P \in C} \mathcal{O}_P = \mathbb{K}$$

Divisor classes and Picard group

- A divisor on C is called **principal** if it is the divisor of a certain $\varphi \in \mathbb{K}(F)^*$, i.e.

$$\text{Prin } C := \{\text{div } \varphi \ : \ \varphi \in \mathbb{K}(F)^*\} \subset \text{Div }^0 C \subset \text{Div } C.$$

- The quotient group

$$\text{Pic } C := \text{Div } C/\text{Prin } C$$

is called the **Picard group** or the group of **divisor classes** of C.

- Two divisors D_1 and D_2 are said **linearly equivalent** if they define the same class, i.e

$$D_1 \sim D_2 \quad \Longleftrightarrow \quad D_1 - D_2 = \text{div } \varphi.$$

Divisor classes and Picard group

Example

If C is a projective line or a conic, then all divisors of $\mathrm{Div}^0 C$ are linearly equivalent, i.e. $\mathrm{Pic}^0 C := \mathrm{Div}^0 C/\mathrm{Prin}\ C$ is trivial.

For curves of higher degree, this is never the case. In fact

Proposition

If $\deg C \geq 3$ then $P \not\sim Q$ for any distinct points $P, Q \in C$. In particular $\mathrm{Pic}^0 C$ is not trivial.

Vector spaces $L(D)$

- Determining "how many" functions there are on a projective curve is an interesting task since one expects that this feature reflects some intrinsic properties of the curve.
- With this aim, we are interested in studying rational function which are regular everywhere but is a finite set of points where we allow poles (or zeros) of a certain order.

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Vector spaces $L(D)$

If $D = \sum_{P \in C} a_P \cdot P$, we define

$$\begin{aligned} L(D) &:= \{\varphi \in \mathbb{K}(C)^* \ : \ \text{div}\ \varphi + D \geq 0\} \cup \{0\} \\ &= \{\varphi \in \mathbb{K}(C)^* \ : \ \text{ord}_P(\varphi) \geq -a_P \text{ for all } P \in C\} \end{aligned}$$

the space of rational functions such that

1) φ **may** have a pole of order at most a_P for all P such that $a_P > 0$,
2) φ **must** have a zero of order at least $-a_P$ for all P such that $a_P < 0$.

Remark

Our aim is to determine $\ell(D) := \dim_{\mathbb{K}} L(D) \in \mathbb{N} \cup \{\infty\}$, which is called **dimension** of the divisor D.

Remark

a) If $D = 0$ the space

$$L(D) = L(0) = \{\varphi \in \mathbb{K}(C)^* \;:\; \text{div } \varphi \geq 0\} \cup \{0\} = \mathbb{K}$$

and so $\ell(0) = 1$.

b) If $\deg D < 0$, we have $L(D) = \{0\}$ and $\ell(D) = 0$.

c) If $D \leq D'$ then $L(D) \subseteq L(D')$ and $\ell(D) \leq \ell(D')$.

d) If $D \sim D'$ are two linearly equivalent divisors on C, then $\ell(D) = \ell(D')$.

Lemma

Let D be a divisor on a projective curve C. Then

(i) for any point $P \in C$ we have $\ell(D+P) = \ell(D)$ or $\ell(D+P) = \ell(D) + 1$.

(ii) For any divisor $D' \geq D$ we have $\ell(D) \leq \ell(D') \leq \ell(D) + \deg(D' - D)$.

(iii) If $\deg D \geq 0$ then $\ell(D) \leq \deg D + 1 < +\infty$.

Lemma

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c) If $\deg D \geq 0$ then $\ell(D) \leq \deg D + 1 < +\infty$.

Proof.

Let $D = \sum_{P \in C} a_P \cdot P$ and consider $\Phi : L(D+P) \ni \varphi \mapsto (t^{a_P+1}\varphi)(P) \in \mathbb{K}$. Observing

$$\Phi(\varphi) = 0 \iff \text{ord}_P(t^{a_P+1}\varphi) > 0 \iff \text{ord}_P(\varphi) + a_P \geq 0$$

we deduce that $\ker \Phi = L(D)$, and $L(D+P)/L(D) \cong Im\,\Phi \subseteq \mathbb{K}$. □

Proposition

If $\deg D \geq 0$ on a projective curve of degree 1 or 2 then $\ell(D) = \deg D + 1$.

Indeed, recall that Pic $C \cong \mathbb{Z}/\text{Pic}^0 C$ and that in this case $\text{Pic}^0 C = \{0\}$.
If P and Q are distinct

$$P \sim Q \quad \Longleftrightarrow \quad \text{div } \varphi = Q - P \quad \Longleftrightarrow \quad \text{div } \varphi^k = kQ - kP$$

and therefore

$$\begin{cases} \varphi^k + kP = kQ \geq 0 \\ \varphi^k + (k-1)P = kQ - P \ngeq 0 \end{cases} \quad \Longrightarrow \quad \varphi^k \in L(kP) - L((k-1)P).$$

We conclude $\mathbb{K} = L(0) \subset L(P) \subset \cdots \subset L(nP)$, and so $\ell(nP) = n + 1$. Since every divisor D of degree n is $D \sim nP$ we conclude.

Proposition

If $\deg C \geq 3$ then

a) for every point $\ell(P) = 1$.

b) if $P \neq Q$, we have $\ell(P - Q) = 0$.

- If $\varphi \in L(P)$, it must have a pole of order 1 in P and be regular elsewhere, and so div $\varphi = Q - P$. By above proposition $Q = P$ and so $L(P) = \mathbb{K}$.
- We have $L(P - Q) \subset L(P) = \mathbb{K}$ and so $L(P - Q) = \{0\}$.

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Proposition

If $\deg C \geq 3$ then

a) for every point $\ell(P) = 1$.

b) if $P \neq Q$, we have $\ell(P - Q) = 0$.

Remark

If $\deg C \geq 3$, for any two distinct points $P, Q \in C$ we have

$$\ell(0) = 1 \quad \text{and} \quad \ell(P - Q) = 0,$$

so in general $\ell(D)$ does not only depend on $\deg D$.

Lemma

If C is a projective curve of degree d (wlog $C \neq Z$). Then for all $n \geq d$ for the divisor $D := n \cdot \text{div } Z$ we have:

a) There is an exact sequence

$$0 \longrightarrow \mathbb{K}[X,Y,Z]_{n-d} \xrightarrow{\cdot C} \mathbb{K}[X,Y,Z]_n \xrightarrow{\div Z^n} L(D) \longrightarrow 0.$$

b) $\ell(D) = \deg D + 1 - \binom{d-1}{2}$.

Riemann's Theorem and degree–genus formula

Theorem (Riemann)

Let C be a smooth, irreducible projective curve of degree d.

- *There is a unique smallest integer g, depending only on C, such that*

$$\ell(D) \geq \deg D + 1 - g, \tag{1}$$

*for any divisor $D \in$ Div C. Such g is called **(algebraic) genus** of C (or of its function field).*

Theorem

If C is a smooth projective plane curve of degree d, its algebraic genus is

$$g = \frac{(d-1)(d-2)}{2}. \tag{2}$$

Proof.

We show that $g := \binom{d-1}{2}$ satisfies $\ell(D) \geq \deg D + 1 - g$ for all D. Notice

- If (1) holds for D it holds for any linearly equivalent $D' \sim D$;
- If (1) holds for D it holds for any $D' \leq D$, in fact

$$\ell(D') \leq \ell(D) - \deg(D - D') \leq \deg D + 1 - g - \deg(D - D') = \deg D' + 1 - g.$$

We can write a divisor on C as $D = P_1 + \cdots + P_n - E$, where E is an effective divisor and $P_i \in C$. Choosing n lines l_i through P_i (not equal to C) we define

$$D' := D + \text{div}\, \frac{Z^n}{l_1 \cdots l_n} \quad \sim D$$

which satisfies

$$D' = P_1 + \cdots + P_n - E + \text{div}\, Z^n - \sum_{i=1}^{n} \text{div}\, l_i \leq \text{div}\, Z^n.$$

A few remarks

- If $\mathbb{K} = \mathbb{C}$ the algebraic genus of a smooth projective plane curve coincides with its topological one.
- We can define the genus of a curve as the genus of the smooth projective curve that is birational to it.
- The genus of a curve is (tautologically) a birational invariant.
- In case C is a projective plane curve of degree d with only ordinary singularities, the "degree–genus formula" must be modified as

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in S} \frac{m_P(m_P - 1)}{2},$$

where S is the set of singular points and m_P the multiplicity of C at P.

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Canonical Divisor

We have proved the following bounds for the dimension of an effective divisor $D \in \text{Div } C$

$$\deg D + 1 - g \leq \ell(D) \leq \deg D + 1.$$

- Notice that we can not expect a formula only depending on $\deg D$.
- It turns out that the difference between $\ell(D)$ and $\deg D + 1 - g$ can be seen as the dimension of another divisor $D' \in \text{Div } C$ related to D.

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Canonical Divisor

Definition

Let C be a projective curve of degree d. For any line l (not equal to C) we define

$$K_C := (d-3)\,\text{div}\ l \qquad \in \text{Pic}\ F$$

the **canonical divisor (class)** of C.

Notice that this definition does not depend on the chosen line since every two lines are linearly equivalent.

Lemma

For any projective curve C of genus g we have $\deg K_C = 2g - 2$.

Proof.

$$\deg K_C = (d-3) = \deg(\operatorname{div} l) = (d-3)d = 2\binom{d-1}{2} - 2 = 2g - 2.$$

$\square$

Lemma

For any point P on a projective curve C we have $\ell(K_C + P) = \ell(K_C)$.

Remark

If $\mathbb{K} = \mathbb{C}$ the latter is a direct consequence of the Residue Theorem for differential forms. Indeed, a form ω can not have exactly one non–zero residue and so for any $\varphi \in \mathbb{K}(C)$

$$\text{div }(\varphi\omega) + P \geq 0 \quad \Longrightarrow \quad \text{div }(\varphi\omega) \geq 0.$$

In other words $L(\text{div }\omega + P) = L(\text{div }\omega)$.

Riemann–Roch theorem

Theorem (Riemann–Roch)

Let C be a smooth projective curve of genus g. Then

$$\ell(D) - \ell(K_C - D) = \deg D + 1 - g$$

for all divisors D on C.

Proof.

- We prove $\ell(D) - \ell(K_C - D) \geq \deg D + 1 - g$ descending induction, using the fact that if $\deg D > 2g - 2$ the thesis follows by Riemann's theorem.
- Induction step: assume the statement holds for D and prove it for $D - P$ for any $P \in C$.

$$\ell(D-P) - \ell(K_C - D + P) \geq \ell(D) - 1 - (\ell(K_C - D) + 1) \geq$$
$$\geq \deg D + 1 - g - 2 = \deg(D - P) - g.$$

- If by contradiction the first inequality is not strict, we can find

$$\varphi \in L(D) - L(D-P) \quad \text{and} \quad \psi \in L(K_C - D + P) - L(K_C - D).$$

But then we have the absurd:

$$\mathrm{div}(\varphi\psi) + K_C + P \geq 0 \text{ with equality at } P.$$

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Remarks

- For the divisor $D = 0$ we have seen $\ell(0) = 1$, and thus

$$1 - \ell(K_C) = 0 + 1 - g \quad \Longrightarrow \quad g = \ell(K_C).$$

Sometimes the latter is taken as the definition of the genus.

- If $\deg D > 2g - 2$, then $\deg(K_C - D) < 0$ and so $\ell(K_C - D) = 0$. Therefore, in this case we have

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In other words, if the divisor is large enough we can compute its dimension just in terms of its degree and the genus of the curve.

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