

UNIVERSITY OF UDINE

Master's degree in Mathematics

Master's Thesis

Singular points in planar dynamical systems and in FLRW metric $f(R)$ cosmology models

Supervisor:

Prof. Stefano Ansoldi

Co–supervisor:

Prof. Sebastiano Sonego

Student:

Nicola Dal Cin

ACADEMIC YEAR 2022 – 2023

Contents

1 Introduction **7**

2 Singular autonomous planar systems **9**
2.1 Autonomous singular planar systems 9
2.1.1 Carathéodory solutions 10
2.1.2 Implicit formulation 11
2.1.3 Regular associated system 12
2.2 Funnel points on the singular curve 13
2.2.1 First case: incident flow 17
2.2.2 Second case: tangential flow 18
2.2.3 Some remarks 19
2.3 Autonomous singular systems 20

3 Metric $f(R)$ theories of gravity **23**
3.1 Historical and experimental motivations 23
3.2 Field equations 25
3.3 Cosmological picture 27
3.3.1 FLRW model for metric $f(R)$ theories 27
3.3.2 Accelerating expansion 29
3.3.3 Cosmological eras 30
3.4 Dolgov–Kawasaki instability 31

4 Singular points in $f(R)$ models **33**
4.1 Stationary points 33
4.1.1 Stability of the stationary points 34
4.2 Singular points 36
4.2.1 Regular associated system 37
4.2.2 Additional condition 39

5 Conclusion **41**

A Equivariant and Reversible Dynamical Systems **43**
A.1 Equivariant and reversible systems 43
A.2 Symmetric periodic orbits 45

B Geometric theory of differential equations **47**
B.1 Implicit differential equations 47
B.2 Singular points 49
B.3 Rational vector fields 50

C Addendum **53**
C.1 Taylor expansion of $\rho(R, H)$ 53

Notations

We summarize here the main notations that will be adopted throughout the present work.

- Vectors, with the only exception of space–time four–vectors, and vector/matrix–valued maps will be written with bold notation.
- We denote the Euclidean scalar product between the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^d$ as

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle := \boldsymbol{v}^T \boldsymbol{w}.$$

- We adopt Newton's notation indicating the time derivative of a function $g(t)$, in symbols:

$$\dot{g}(t) := \frac{dg}{dt}(t).$$

- To denote derivatives with respect to dynamical variables different from t, we will also adopt the lower index notation, i.e. $f_R(R) := \frac{df}{dR}(R)$. This notation will also be used for partial derivatives.
- To represent tensors, we use the *abstract–index* notation, see [Wal10].
- We assume to work with speed of light $c = 1$, and denote $\kappa := 8\pi G$, where G is the gravitation constant.

Chapter 1

Introduction

In this work, we address the problem of studying the qualitative behaviour of Carathéodory solutions[1] of a planar autonomous system of Ordinary Differential Equations (ODEs), around singular points, i.e. points where the vector field is not well defined. In particular, we will consider the case in which the right–hand side term is a rational function obtained by the quotient of two C^1 maps:

$$\begin{cases} \alpha(x,y)\ \dot{x} = \beta(x,y) \\ \gamma(x,y)\ \dot{y} = \delta(x,y), \end{cases} \tag{1.1}$$

and the behaviour of the solutions can be deduced by studying a continuous dynamical system associated with the singular one (1.1).

This approach stands out with respect to the analytical techniques which are usually employed in the study of singularities, such as finding series solutions in a neighborhood of the singularity [Bir09], or applying a suitable change of coordinates to solve the point by a σ–process [Arn12], i.e. by blow–up. As we shall see, this viewpoint emphasizes the geometric main feature of singular points, which indeed correspond to regions where the integral curves of the vector field are not parallel to each other.

Moreover, we will address the case of a particular class of singular points: the *funnel singular points*, that attract all the solutions originating in an open set V_1, which after reaching the singularity are propagated into another open set V_2. In particular, we will provide sufficient conditions for a singular point to be a funnel (which are also sufficient when it is non–degenerate) and also address the problem of continuing Carathéodory solutions past a funnel point.

This mathematical analysis is motivated by new physical, demographic, and engineering models (see [RB03], [CD89], [BS19]) where the time–derivative of the dynamical variables is multiplied by a matrix depending on them. In Chapter 2, we focus on the planar case and in particular on funnel points — as the main application for our physical example; nevertheless, as will be suggested in the next chapters, other interesting features can be studied. For instance, in the present work, we mainly restrict ourselves to the two–dimensional case, but the problem can be contextualized, and the definitions can be extended to the generic d–dimensional case (as hinted in Section 2.3).

In Chapter 3, we briefly introduce the historical and physical motivations supporting metric $f(R)$ modified theories of gravitation. The latter, are obtained by considering a more general dependence with respect to the Ricci scalar curvature R in the gravitational Lagrangian which, in the case of General Relativity (GR), depends only linearly on R. Moreover, in this part, we introduce the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological model for metric

[1] In other words, absolutely continuous curves which satisfy the differential equation for almost all points in their time domain.

$f(R)$ theories of gravity, and discuss some physical motivations to restrict to the case of convex functions $f(R) \in C^3(I)$ with $f_{RR} > 0$ in the open domain $I \subseteq \mathbb{R}$ — in order to avoid the Dolgov–Kawasaki instability.

In Chapter 4, we apply the general results for systems of the form (1.1) investigated in Chapter 2, to study the phase space and the singular points of a spatially–flat FLRW cosmological model for a metric $f(R)$ theory of gravity. Indeed, in the matter–free and spatially flat case, the latter reduces to studying a singular system:

$$\begin{cases} \dot{R} &= \dfrac{1}{f_{RR}(R)H}\left[\dfrac{1}{6}\Big(f_R(R)R - f(R)\Big) - f_R(R)H^2\right], \\ \dot{H} &= R/6 - 2H^2, \end{cases} \tag{1.2}$$

where f is a fixed function of the Ricci scalar curvature R, and the other dynamical variable is the Hubble parameter H. In particular, we will prove that if a singular point (R_s, H_s) of (1.2) is not the origin and $f_{RR}(R_s) \neq 0$, it is a nondegenerate funnel point. A solution approaching the singular point, since the latter lies in the R–axis (i.e. $H_s = 0$), is a *bouncing cosmological solution* [SZ18]. In other words, it represents a universe that is contracting till it reaches the singularity after which it starts to expand (or vice–versa).

Finally, also referring to Chapter 2 and to the Geometric Theory of ODEs, we conclude our dissertation by discussing additional conditions to continue a Carathéodory (bouncing) solution past a singular point. In particular, we examine the scenario where we prescribe the continuation of the solution while maintaining the incoming direction at the singular point. In the case of nondegenerate funnel points, this condition enables the recovery of uniqueness for Initial Value Problems (IVPs); consequently, it becomes feasible to define a continuous dynamical system in which (1.2) is well–defined.

Chapter 2

Singular autonomous planar systems

In recent years, there has been an increase of interest in differential equations that do not satisfy the usual regularity properties, and for which the classical methods of the theory of continuous dynamical systems are not always valid. Some of the applications which motivate such an interest come from engineering (see e.g. [RB03], [CD89]), demographic [BS19], or physical models.

As it is well known, when such systems are considered, Peano and Picard–Lindelöf theorems [Hal09] for the existence and uniqueness of solutions are likely to fail. Furthermore, one is often suggested to introduce other weaker notions of solutions for these types of equations — e.g. Carathéodory solutions.

A particular class of such differential equations is the one having a discontinuous right–hand side, as it happens in numerous mechanical models where phase space is divided into regions where a different vector field is defined. The Russian mathematician Filippov studied such equations in depth, and adopting the theory of differential inclusions introduced a suitable definition of solution which has found many applications also in control problems (see [Cor08] and references therein). An exhaustive classification of the phase portrait of Filippov solutions of Discontinuous Differential Equations (DDE[1]) has been made; however, in this context, the vector field is assumed to have a finite limit at the discontinuity singularity [Fil13].

Nevertheless, more and more examples of differential equations with discontinuous and *unbounded* right–hand side (like a rational function) are being proposed. One in general has to deal with equations of the form

$$\boldsymbol{F_1}(\boldsymbol{x})\ \dot{\boldsymbol{x}} = \boldsymbol{F_2}(\boldsymbol{x}), \tag{2.1}$$

with $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^d$, where $\boldsymbol{F_1} : \Omega \to \mathbb{R}^{d\times d}$ and $\boldsymbol{F_2} : \Omega \to \mathbb{R}^d$ are fields which can vanish in some subsets of the domain.

Motivated by some peculiar behaviour observed in cosmological examples coming from a class of modified theories of General Relativity — see Section 4.2.1 — we address our attention to two–dimensional singular systems.

2.1 Autonomous singular planar systems

In this section, we will use notations and methods of the geometric theory of differential equations, which are briefly recalled in Appendix B, whose reading is recommended to the reader.

Let us consider an autonomous, two–dimensional system of differential equations of the form

$$\begin{cases} \alpha(x,y)\ \dot{x} = \beta(x,y) \\ \gamma(x,y)\ \dot{y} = \delta(x,y), \end{cases} \tag{2.2}$$

[1]The acronym DDE might be confused with the one for Delayed Differential Equations, which however are not treated in the present work, leading to no ambiguity.

that can be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}), \quad \text{letting} \quad \boldsymbol{F}(\boldsymbol{x}) := \left(\frac{\beta(x,y)}{\alpha(x,y)}, \quad \frac{\delta(x,y)}{\gamma(x,y)} \right) \tag{2.3}$$

where $\alpha, \beta, \gamma, \delta \in C^1(\Omega)$, and Ω is an open subset of $\mathbb{R}^2$. In the notations of (2.1), this corresponds to choosing $d = 2$ and

$$\boldsymbol{F_1}(\boldsymbol{x}) := \begin{pmatrix} \alpha(\boldsymbol{x}) & 0 \\ 0 & \gamma(\boldsymbol{x}) \end{pmatrix}, \quad \boldsymbol{F_2}(\boldsymbol{x}) := \begin{pmatrix} \beta(\boldsymbol{x}) \\ \delta(\boldsymbol{x}) \end{pmatrix}. \tag{2.4}$$

The above assumptions are made in order to avoid problems in defining a unique (classical) solution away from the *singular curve*:

$$\mathtt{SC} := \{(x,y) \in \Omega \mid \alpha(x,y)\gamma(x,y) = 0\}. \tag{2.5}$$

Nevertheless, it will be interesting to analyze a few notable qualitative properties of the solution close to the singularity, as there may be a loss of uniqueness at SC, and other pathological behaviours that lead one to consider weaker definitions of solutions.

2.1.1 Carathéodory solutions

Before treating the main results of this section, a clarification on the notion of *solution* of the differential equation (2.2) is needed.

Example 1. Consider the following singular system of ODEs:

$$\begin{cases} \dot{x} = -x/y \\ \dot{y} = -1. \end{cases} \tag{2.6}$$

As the second component of the vector field $F_2(\boldsymbol{x}) \equiv -1$ is always negative, the solutions

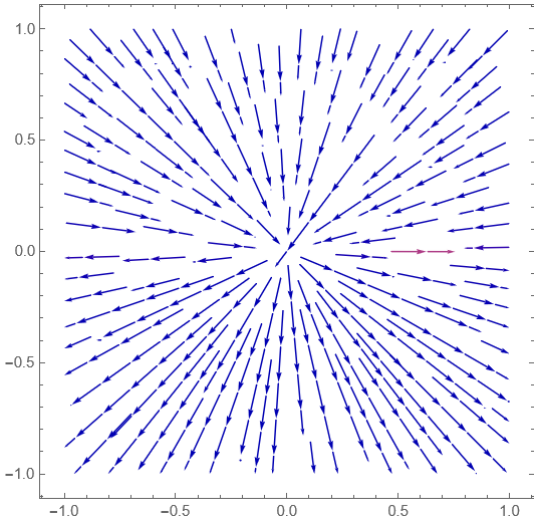

Figure 2.1: Phase portrait of the singular system (2.6).

reach the singular curve $y = 0$ in a finite time. Moreover, notice that it is possible for two different integral curves to reach the same point in finite time (see Figure 2.1), hence the natural problem of continuing the solution past the singularity arises. For this reason, we will consider Carathéodory solutions of (2.2), i.e. absolutely continuous curves that satisfy the differential equations for Lebesgue–a.e. times in their time–domain.

Later, we will also examine the possibility of including an additional prescription at the singular point, in order to continue the solution in a natural (e.g. C^1–continuous) way. For instance, in the above Example 1 a natural prescription seems to be to "go straight in the phase plane", as

$$\ddot{x} = -\frac{-\frac{x}{y}\cdot y + x\cdot 1}{y^2} \equiv 0, \tag{2.7}$$

and so x/y is a constant of the motion.

2.1.2 Implicit formulation

As usual, the integral curves of (2.2) can be seen as a parametrization of the solutions of the ODE:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\alpha(x,y)\delta(x,y)}{\beta(x,y)\gamma(x,y)}, \tag{2.8}$$

where the right hand side is defined, i.e. out of SC. Therefore, one could take a step forward and try to apply some geometric methods of [Arn12] to study the implicit differential equation

$$G(x,y,p) = \beta(x,y)\gamma(x,y)\,p - \alpha(x,y)\delta(x,y) = 0. \tag{2.9}$$

We recall that the latter defines a surface in the three–dimensional space of 1–jets with coordinates (x,y,p), which is smooth where the total differential of G is nonzero (contrary to Arnold we let this happen in a finite number of points).

The *singular points*, i.e. those points of the surface $G=0$ at which $\partial G/\partial p = 0$, form the so called *criminant curve*, defined as

$$\texttt{CC} := \{(x,y,p) \mid \alpha(x,y)\delta(x,y) = 0 \quad \text{and} \quad \beta(x,y)\gamma(x,y) = 0\}. \tag{2.10}$$

Notice that in the generic case in which the above two conditions are independent, the criminant curve is composed by vertical lines, whose projection on the (x,y)–plane are isolated points. In other terms, in general, the *discriminant curve* of (2.8) is composed by isolated points, which with some abuse of notation we will still call *singular points* of (2.8) and (2.2).

Remark 1. The definition above matches with one's expectations, indeed, singular points (as defined above) correspond to points where the phase curves of the vector field in (2.2) are not (locally) parallel to each other. Furthermore, notice that such points not only include the equilibria of the singular system (2.2) — much studied in the theory of continuous dynamical systems and relevant for the dynamics of a system as important results like the Poincaré–Bendixon theorem — but also other points, that we will call *proper singular points* of (2.2). The latter, are not as much studied as the equilibria (since they are not present in non–singular systems), but they are likewise important from a dynamical point of view.

Furthermore, notice that the derivative of the map

$$\Phi : (x,y,p) \mapsto (G, G_p) \tag{2.11}$$

is

$$D\Phi = \begin{pmatrix} \frac{\partial(\beta\gamma)}{\partial x}p - \frac{\partial(\alpha\delta)}{\partial x} & \frac{\partial(\beta\gamma)}{\partial y}p - \frac{\partial(\alpha\delta)}{\partial y} & \beta\gamma \\ \\ \frac{\partial(\beta\gamma)}{\partial x} & \frac{\partial(\beta\gamma)}{\partial y} & 0 \end{pmatrix}, \tag{2.12}$$

which has maximal rank if and only if the derivative of the vector field

$$\tilde{\boldsymbol{F}}(x,y) := \big(\alpha(x,y)\delta(x,y), \quad \beta(x,y)\gamma(x,y)\big) \tag{2.13}$$

has rank 2 — since $\beta\gamma = 0$ at singular points. In other words, the regular singular points of (2.3) (in the sense of Definition B.2.2) are equilibrium points (x_e, y_e) of

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{F}}(\boldsymbol{x}) \tag{2.14}$$

such that $\det[D\tilde{\boldsymbol{F}}(x_e, y_e)] \neq 0$.

2.1.3 Regular associated system

Following the above reasoning, it is natural to associate to (2.2) the system (2.14) which shares the same phase portrait, and so singular points, but which is not singular in the sense that has a $C^1(\Omega)$ vector field. We therefore call the dynamical system defined by

$$\begin{cases} \dot{x} = \beta(x, y)\gamma(x, y) \\ \dot{y} = \delta(x, y)\alpha(x, y), \end{cases} \tag{2.15}$$

the *regular dynamical system associated with* (2.2). In other words we consider $\dot{\boldsymbol{x}} = \tilde{\boldsymbol{F}}(\boldsymbol{x})$ with the vector field defined in (2.13).

Notice that the two systems (2.2) and (2.15) have a common phase portrait outside SC (since they have the same $p = dy/dx = \dot{y}/\dot{x}$ there), but the integral curves of the two are different, as the parametrization of the curves in phase space is not alike.

To summarize, the main intent of this procedure is to employ the classical, powerful tools of the theory of continuous dynamical systems — applicable to the regular system (2.15) — to deduce the properties of (proper) singular points of (2.2), and the dynamics in a neighborhood of them. We point out that this can be done for autonomous systems of ODEs in any dimension, see Section 2.3.

More in detail, it is immediate to see that the time derivative of the components of the solution of (2.2) and (2.15) have the same sign if $\alpha\gamma > 0$, and the opposite one if $\alpha\gamma < 0$. Therefore, one can deduce the qualitative dynamics (i.e. the direction in which the phase curves of (2.2) are parametrized) inverting the "time arrows" in the phase curves of the system (2.15) where $\alpha\gamma < 0$. In other terms, the singular curve

$$\mathtt{SC} := \{(x, y) \in \Omega \mid \alpha(x, y)\gamma(x, y) = 0\} \tag{2.16}$$

divides the phase space into connected components, where the dynamics is deducible by the one of the regular associated system and by the sign of $\alpha(x, y)\gamma(x, y)$.

In particular, we expect to observe interesting behaviours when the phase curves are tangent to the singular curve in a neighborhood of a singular point, or when SC belongs to the fixed sub–manifold with respect to some symmetry — e.g. the H–axis in the system (1.2).

As mentioned, Filippov already studied the phase portraits of planar differential equations with discontinuous right–hand side [Fil13], but — to the best of our knowledge — he always considered vector fields with finite limit at the singularity. Therefore, in general, we expect to deal with a different analysis and results.

Furthermore, by using the regular associated system it is possible to define proper singular points in an easy and generalizable way:

Definition 2.1.1.

- A point $\boldsymbol{x}_s \in \Omega$ is a *proper singular point* of (2.2), if it is an equilibrium point of the regular associated system (2.15), but not of (2.2).
- A *nondegenerate (proper) singular point* of of (2.2) is a proper singular point $\boldsymbol{x}_s \in \Omega$ such that

$$\det[D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)] \neq 0. \tag{2.17}$$

Example 2. We continue Example 1 by noting that in this case the unique (proper) singular point is the origin, while the regular associated system (which is always determined up to a multiplicative factor) is

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y. \end{cases} \tag{2.18}$$

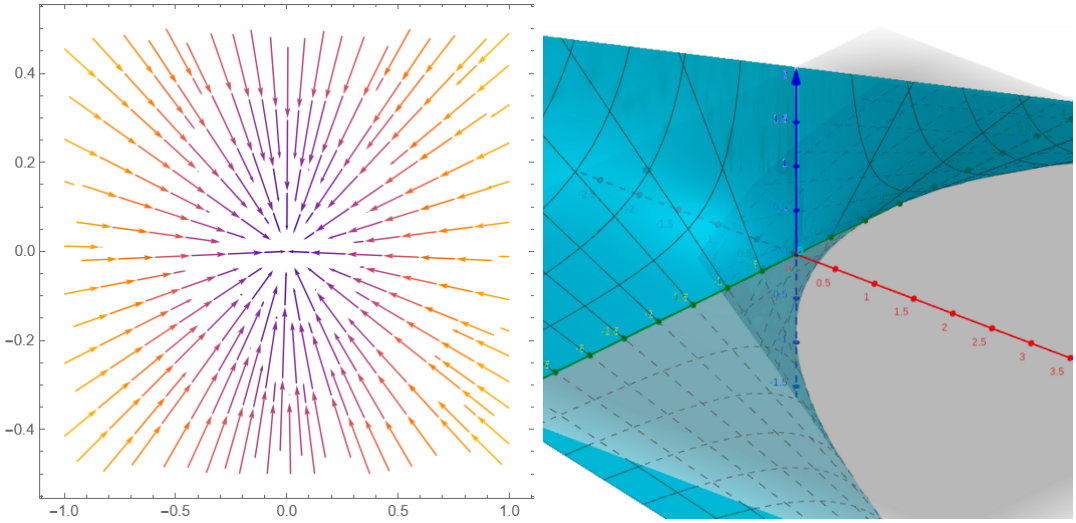

Figure 2.2: The regular associated system to (2.6), and the surface $p = dy/dx = y/x$ in the space of 1–jets. The criminant curve is the vertical p–axis, resulting in a unique projected singular point on the (x, y)–plane: the origin.

2.2 Funnel points on the singular curve

From now on, as we are mainly interested in the local behaviour of a singular system, we will assume that the only component of the vector field which can blow–up is the first one. With this assumption the system (2.2) reduces to

$$\begin{cases} \dot{x} = \dfrac{\beta(x, y)}{\alpha(x, y)} \\ \dot{y} = \delta(x, y), \end{cases} \tag{2.19}$$

and the singular curve

$$\mathtt{SC} := \{(x, y) \in \Omega \mid \alpha(x, y) = 0\}, \tag{2.20}$$

will be assumed smooth — i.e. such that $\boldsymbol{\nabla}\alpha$ does not vanish on the curve. Moreover, we assume to work in a generic case where $\alpha = 0$, $\beta = 0$ and $\delta = 0$ do not intersect simultaneously in the domain Ω — a case that can still be handled separately. Then, provided we restrict the open set Ω, the function $\delta(x, y)$ is non–zero there.

Finally, we assume that the points on the singular curve where the vector field can be extended with a (non unique!) finite value are discrete; in other words, the curves $\alpha = 0$ and $\beta = 0$ have only a discrete set of singular points of intersection.

We are now interested in determining the qualitative behaviour of Carathéodory solutions of (2.19) in a neighborhood of the singular curve $\alpha = 0$ not containing singular points. This is

related to how the surface $z = \alpha(x, y)$ intersects the (x, y)–plane, indeed, in the generic case, the intersection between the surface and the plane $z = 0$, i.e. the singular curve, divides two regions where $\alpha > 0$ and $\alpha < 0$, as in Figure 2.3.

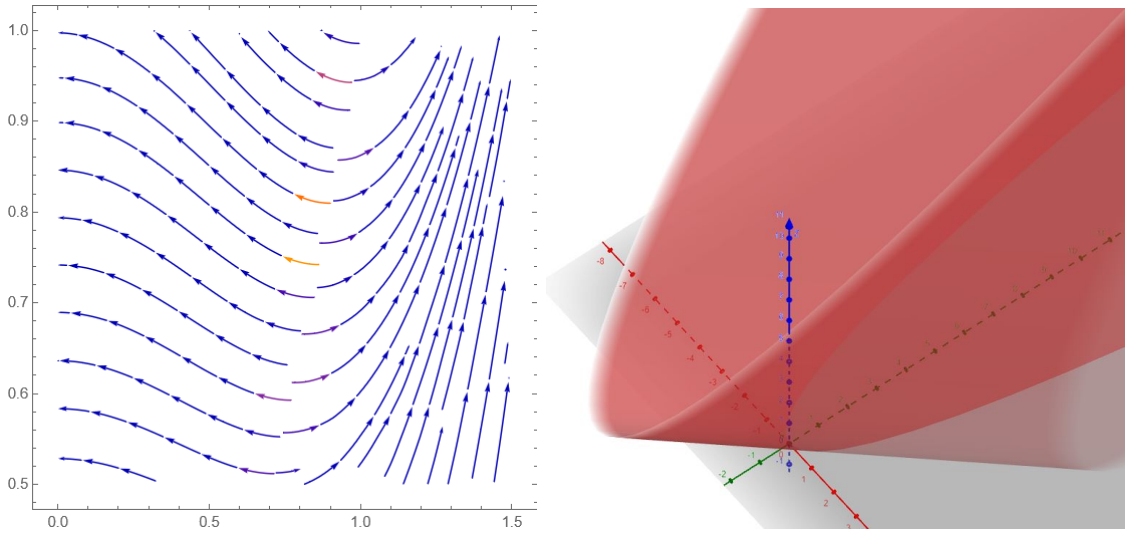

Figure 2.3: On the left there is the phase portrait of $\dot{x} = x/(x^2 - y)$, $\dot{y} = x^2$, while on the other hand the surface $z = \alpha(x, y)$ which intersects the (x, y)–plane in the singular curve.

Notice that, in this generic case, a Carathéothory solution can not evolve through a non–singular point of the singular curve. Indeed, assume that a solution $\boldsymbol{x}(t)$ of (2.19) reaches the point $\bar{\boldsymbol{x}} \in \mathtt{SC}$ at time $\bar{t}$. If this point is not singular, i.e. if $\beta(\bar{\boldsymbol{x}}) \neq 0$ then, depending on the sign of the latter, we have that

$$\lim_{t \to \bar{t}^-} \frac{dx(t)}{dy(t)} = \lim_{t \to \bar{t}^-} \frac{\beta(\boldsymbol{x}(t))}{\alpha(\boldsymbol{x}(t))\delta(\boldsymbol{x}(t))} = \pm\infty. \tag{2.21}$$

However, as the sign of α changes when crossing the singular curve, we have that

$$\lim_{t \to \bar{t}^+} \frac{dx(t)}{dy(t)} = \lim_{t \to \bar{t}^+} \frac{\beta(\boldsymbol{x}(t))}{\alpha(\boldsymbol{x}(t))\delta(\boldsymbol{x}(t))} = \mp\infty. \tag{2.22}$$

Now, two different cases can occur. If the singular curve does not coincide with a line parallel to the x–axis in Ω, then the solutions arriving from the region

$$\Omega_- = \{\boldsymbol{x} \in \Omega \mid \alpha(\boldsymbol{x}) < 0\} \tag{2.23}$$

have opposite tangent vector field with respect to those coming from

$$\Omega_+ = \{\boldsymbol{x} \in \Omega \mid \alpha(\boldsymbol{x}) > 0\}, \tag{2.24}$$

leaving no chance of continuing the solution (see Figure 2.3 left). Otherwise, if the singular curve is the x–axis in Ω, or a line parallel to it, we can consider the associated regular system

$$\begin{cases} \dot{x} = \beta(x, y) \\ \dot{y} = \alpha(x, y)\delta(x, y). \end{cases} \tag{2.25}$$

The conclusion, in this case, follows by the fact that the orbits as (2.19) are the same of (2.25), and in the latter, a phase curve cannot cross the singular curve SC — which contains solutions of (2.25) having $y =$ constant — else the uniqueness of solutions would be violated.

Remark 2. From the above discussion, one may deduce that a Carathéodory solution of systems of the form (2.19) never evolves through a non–singular point $\boldsymbol{x}$ of SC. However this is true only in the generic case when the surface $z = \alpha(x, y)$ properly intersects the plane $z = 0$ at $\boldsymbol{x}$ (see the example in Figure 2.4), in other words when SC separates two regions where α assumes the opposite sign, or when some symmetry is present — e.g. when the singular curve is a line parallel to the x–axis as in the case we will tackle later.

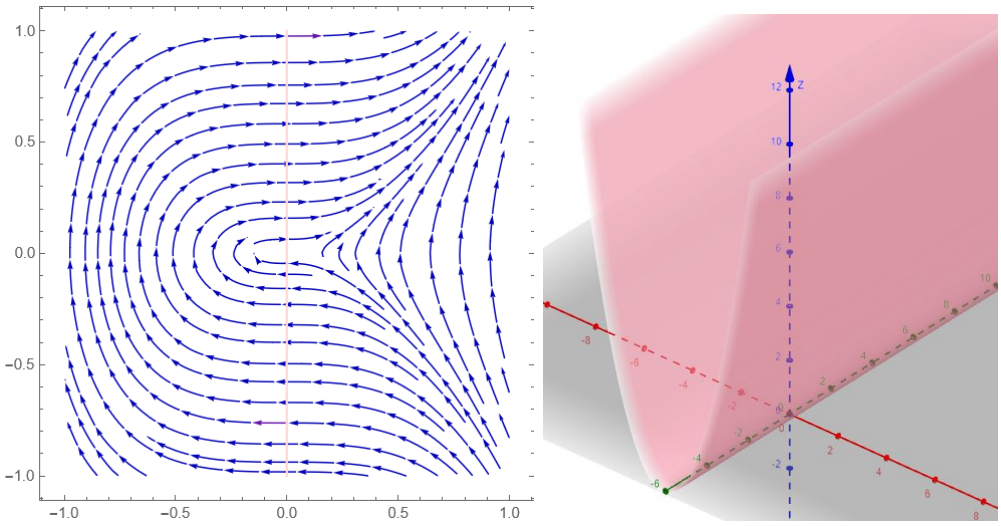

Figure 2.4: The phase portrait of the system $\dot{x} = y/x^2$, $\dot{x} = 1 + x^2$ and the surface $z = x^2$ which is tangent to the (x, y)–plane on the singular curve $x = 0$. In this case, SC does not divide two regions where α has the opposite sign as in the generic case, and so the Carathéodory solutions of the system can evolve through its non–singular points.

Motivated by the discussion above and by the physical models we will study in the next Chapters, from now on we assume that the singular curve is a line parallel to the x–axis. In other words, we assume the *non–transversality condition*

$$\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) \equiv 0 \quad \text{in} \quad \mathtt{SC}, \tag{2.26}$$

as this implies

$$\langle \tilde{\boldsymbol{F}}(\boldsymbol{x}), \boldsymbol{\nabla}\alpha(\boldsymbol{x})\rangle = \beta(\boldsymbol{x})\frac{\partial \alpha}{\partial x}(\boldsymbol{x}) = 0, \tag{2.27}$$

i.e., up to a coordinate change, the singular curve is the x–axis in Ω.

This choice will be clearer later, as we will be interested in dynamics in a neighborhood of a singular point $\boldsymbol{x}_e$ which corresponds to a node of the associated dynamical system. With this in mind, the above assumption corresponds to asking that the singular curve coincides with a separatrix of the node.

Proposition 1. *Assume $\alpha = 0$ and $\beta = 0$ have a discrete intersection, and they do not cross $\delta = 0$ simultaneously. If the non–transversality condition holds, a Carathéodory solution of (2.19) can evolve through the singular curve $\alpha = 0$, if and only if it passes through a singular point.*

Proof. As discussed above, it follows by the fact that the phase space of (2.19) is the same as the regular associated system

$$\begin{cases} \dot{x} = \beta(x, y) \\ \dot{y} = \alpha(x, y)\delta(x, y), \end{cases} \tag{2.28}$$

and that the singular curve SC which contains solutions of (2.28) having $y =$ costant and that tend to/escape by the equilibrium (x_s, y_s). □

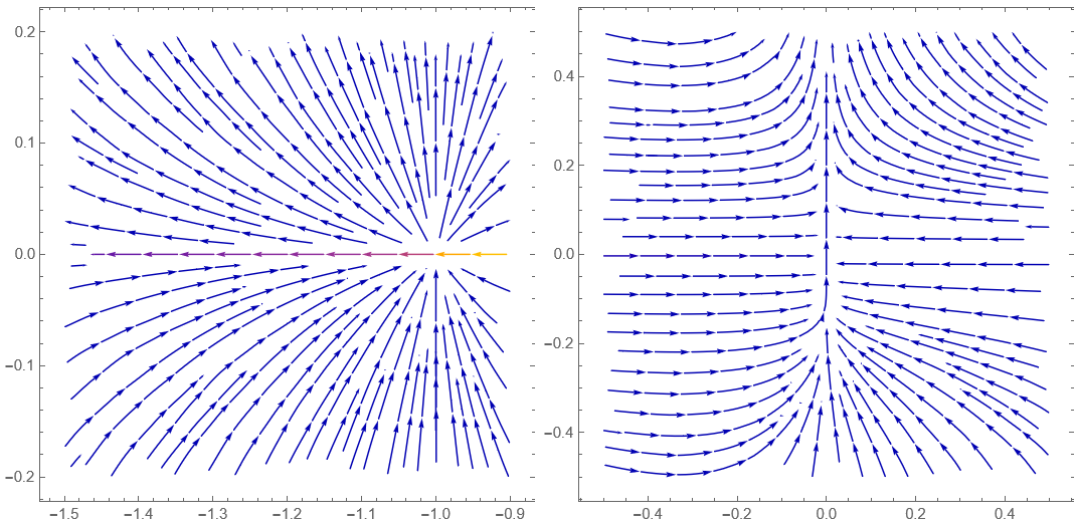

Figure 2.5: A generic and a non–generic example where Carathéodory solutions evolve though a singular–point in SC when (2.26) holds. On the left: the phase portrait of $\dot{x} = (1+x)/y$, $\dot{y} = x^2$. On the right: the one of the system $\dot{x} = -x/y^2$, $\dot{y} = 1 + 3x$.

To describe the qualitative behaviour of solutions in a neighborhood of a singular point, as in the figure above, we give the following definition.

Definition 2.2.1. A point $\bar{\boldsymbol{x}}$ on the singular curve is called a *funnel point* if there is an open neighborhood V of $\bar{\boldsymbol{x}}$ such that

1) no Carathéodory solution of (2.19) with initial data in V can cross the singular curve if not passing through $\bar{\boldsymbol{x}}$.

2) In addition,

   i) all the Carathéodory solutions of (2.19) originating in

$$V_+ := V \cap \Omega_+ \tag{2.29}$$

   contain such a point in the future [respectively in the past];

   ii) all the Carathéodory solutions of (2.19) originating in

$$V_- := V \cap \Omega_- \tag{2.30}$$

   contain such a point in the past [respectively in the future].

Notice that by the proposition above, under our assumptions, a funnel point must be a singular point. In the following result, we exploit the correspondence with a regular system associated with (2.19) in order to find sufficient conditions for a singular point to be a funnel point in the generic case.

Theorem 1. *Assume $\alpha = 0$ and $\beta = 0$ have a discrete intersection, and they do not cross $\delta = 0$ simultaneously. Suppose that the singular system* (2.19) *has a singular point $\boldsymbol{x}_s$ on SC and*

that $z = \alpha(x, y)$ properly[2] *intersects the (x, y)–plane on the x–line, i.e. the non–transversality condition* (2.26) *holds in Ω. If $\boldsymbol{x}_s$ corresponds either to a (locally) asymptotically stable/unstable node of the regular system* (2.25)*, then the singular point $\boldsymbol{x}_s$ is a funnel point.*

Proof. The idea of the proof is to consider the regular associated system (2.25), and then transfer the properties of its orbits to those of (2.19). Let us assume that $\boldsymbol{x}_s$ is a locally asymptotically stable equilibrium of (2.25), as the other case is analogous. Then there exists a neighborhood $U \subseteq \Omega$ of $\boldsymbol{x}_s$ and an open set $V \subseteq U$ neighborhood of $\boldsymbol{x}_s$ such that any (classical) solution $\tilde{\boldsymbol{x}}(t; \boldsymbol{x}_0)$ of (2.25), originating in $\boldsymbol{x}_0 \in V$, stays in U in future times, i.e.

$$\tilde{\boldsymbol{x}}(0; \boldsymbol{x}_0) = \boldsymbol{x}_0 \in V, \quad \text{and} \quad \tilde{\boldsymbol{x}}(t; \boldsymbol{x}_0) \in U \quad \text{for all} \quad t \geq 0. \tag{2.31}$$

Moreover, as the equilibrium is asymptotically stable, we have that all solutions in V tend to $\boldsymbol{x}_s$ as $t \to +\infty$ (of course without reaching it in finite time, to avoid the violation of the uniqueness of solutions). The main idea is that in this case the associated system has solutions with constant y–component which can not be crossed by other phase curves. Also notice that the integral curves of the system (2.19) have the same time direction as the one of (2.25) in V_+, and the opposite one in V_-.

In addition, we assumed that δ does not vanish in $\boldsymbol{x}_s$, hence its sign is constant in the open set V (up to a restriction), hence the solution crosses the singularity in a finite time and points (2.i) and (2.ii) of the above definition hold. Finally, point (1) of the definition follows by the above Proposition.

□

Remark 3.

- For simplicity the theorem above is stated with respect to the reduced form (2.19). Notice however that in a more general case, when the singular curve coincides with the separatrix of the node, one can reduce to (2.19) by a coordinate change.
- By the principle of linearized stability it follows that the last condition, in the above statement, holds if all the eigenvalues of $D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)$ have negative (or positive) real parts.

In order to study the dynamics in a neighborhood of a singular point, usually a process of linearization or Taylor expansion of the particular vector field has to be employed.

However, if the singular curve is a line parallel to the x–axis and the singular point is regular, two qualitatively different local behaviours of the phase curves might take place at a funnel point (which in this case correspond to a node of the regular associated system). Indeed, in this case the integral curves reach the singular point either with different slopes in the phase plane, or they are all tangent to each other at the singular point $\boldsymbol{x}_s$.

2.2.1 First case: incident flow

The first possibility is that the integral curves have different slope in phase space at the singular point. If the latter is a nondegenerate singular point $\boldsymbol{x}_s$, this happens if and only if the Jacobian matrix of the regular associated vector field $D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)$ has the same two non–zero real eigenvalues with geometric multiplicity 2, as phase space is locally topologically equivalent to the normal form discussed in Example 2 (see Figure 2.6 (left)).

Although it is less generic, this case is actually the more intriguing, as there is a unique C^1 continuation of a Carathéodory solution at the singularity. Indeed, in this case the condition of keeping the incoming direction in phase space at the singularity, uniquely determines a single,

[2] In other words, SC separates two regions where α assumes the opposite sign (contrary on what happens in the example in Figure (2.4)).

smooth integral curve in a neighborhood of $\boldsymbol{x}_s$. Hence, with the latter, we recover the existence and uniqueness of C^1 solutions of (2.19) with initial data in an open set $V_* = V - \mathtt{SC}$.

Furthermore, notice that if the implicit differential equation surface (2.9) in the space of 1–jets is not singular at the points with coordinates (x_s, y_s, p), this condition corresponds to the geometric one of continuing the solution curve smoothly on the surface (see Figure 2.6 (right)).

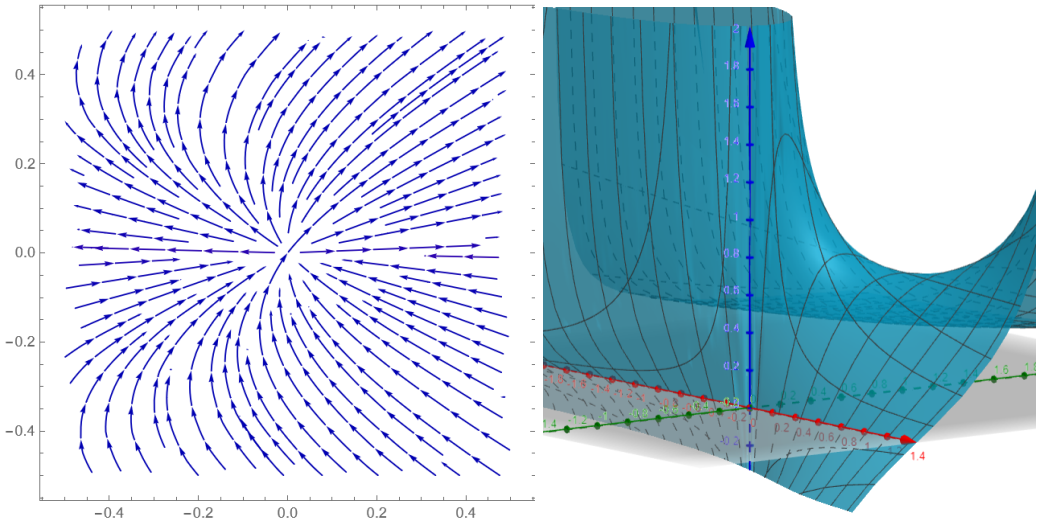

Figure 2.6: One the left, the phase portrait of the singular system $\dot{x} = \frac{x-2(x-1)y^2}{y}$, $\dot{y} = 1$ around the origin, which is a nondegenerate singular point. In this case the eigenvalues of $D\tilde{\boldsymbol{F}}(0)$ are $\lambda_1 = \lambda_2 = 1$. On the right: the correspondent phase surface obtained lifting the phase plane in the (x, y, p)–space of 1–jets.

2.2.2 Second case: tangential flow

In the generic case, however, the solutions are tangent to each other at the singular point $\boldsymbol{x}_s$, as in the examples in Figures 2.7 and 2.8. If such a point is regular, this correspond to the case of $D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)$ having two different, non–zero, real eigenvalues of the same sign (otherwise the associated equilibrium would be a saddle), or the same real eigenvalue $\lambda_1 = \lambda_2$ but with geometric multiplicity 1 — see [Sol77]. Hence, there are more possibilities leading to to this scenario, which is more frequent than the one described above.

In this case, the additional prescription of preserving the tangential direction at $\boldsymbol{x}_s$ does not identify a unique continuation of a Carathéodory solution at the singularity. This prevents from recovering the uniqueness of solution, unless some condition on higher derivatives $d^k y/dx^k$, $k > 1$ is introduced, which can still be motivated by looking at the solution surface in the space of k–jets.

It is interesting to notice that if one considers a (weaker) condition such as imposing the solution to be C^1, Carathéodory solutions can no longer always be continued arbitrarily after the singularity. In fact, an incoming Carathéodory solution has to be continued in a proper way at the singularity, in order to preserve its incoming direction and not to have a cusp and lose the C^1–continuity. For instance, if all the solutions approach the singular point with direction parallel to the singular line (as in Figure 2.8), and there is another *separatrix curve* passing though $\boldsymbol{x}_s$, it divides the accessible and the non–accessible region for an incoming solution.

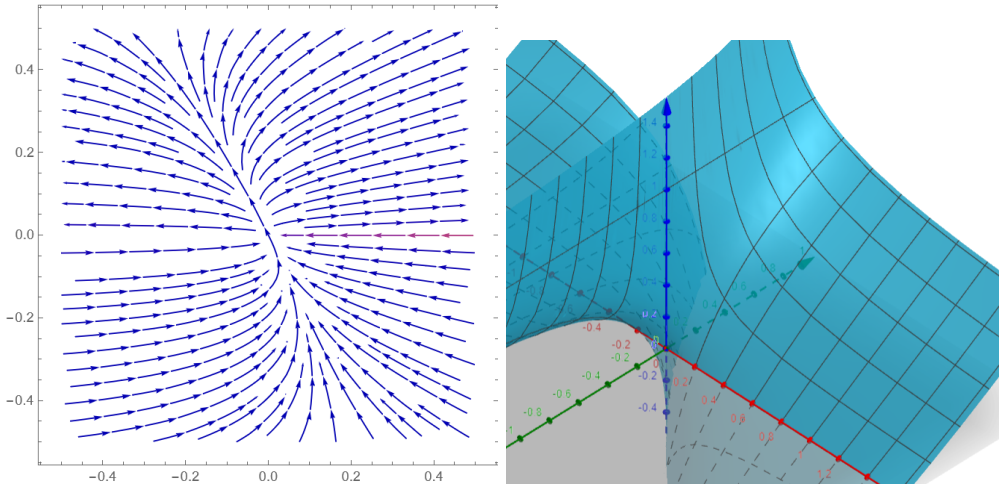

Figure 2.7: One the left, the phase portrait of the singular system $\dot{x} = \frac{3x-(x-1)y}{y}$, $\dot{y} = 1 + x$ around the origin, which is a nondegenerate singular point. In this case the eigenvalues of $D\tilde{\boldsymbol{F}}(0)$ are $\lambda_1 = (3-\sqrt{5})/2$ and $\lambda_2 = (3+\sqrt{5})/2$. On the right: the correspondent phase surface obtained lifting the phase plane in the (x, y, p)–space of 1–jets.

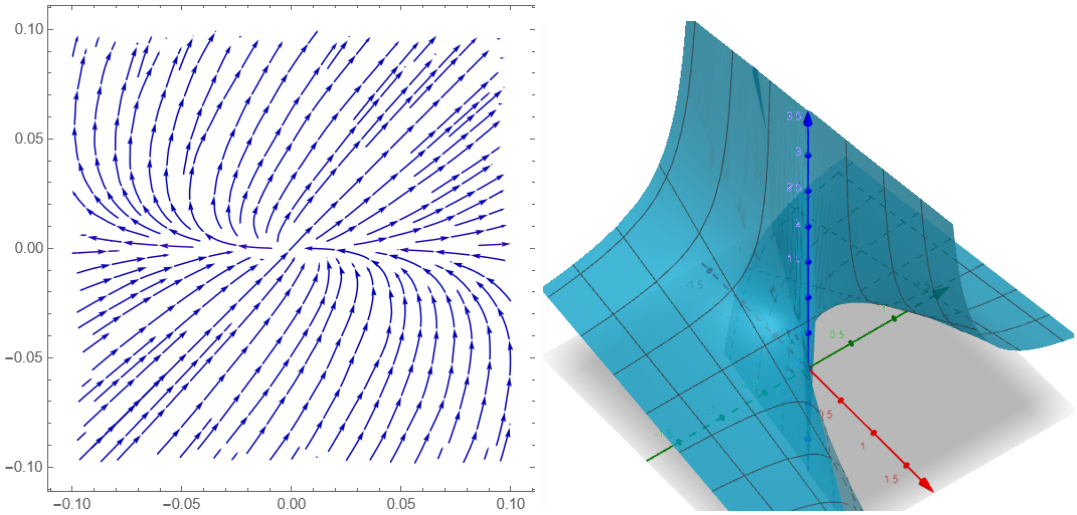

Figure 2.8: On the left: the phase portrait of the singular system $\dot{x} = \frac{x+2y}{3y}$, $\dot{y} = 1$ around the origin, which is a nondegenerate singular point. In this case the eigenvalues of $D\tilde{\boldsymbol{F}}(0)$ are $\lambda_1 = 1$ and $\lambda_2 = 3$, while the other separatrix is the bisector line of the first quadrant, i.e. $y = x$. On the right: the correspondent phase surface obtained lifting the phase plane in the (x, y, p)–space of 1–jets.

2.2.3 Some remarks

Singular points such as those discussed above are often present in many applications and models (as the one we will present in the next chapters); however, they are not always well studied and understood, as the authors often restrict their analysis to an open set not containing them. However, such points can be really important for the dynamics, as the behaviour of a solution in a neighborhood of them can influence drastically the subsequent evolution, and the passage at such a point is sometimes obliged (since they correspond to equilibria with a certain basin of attraction after all!).

These peculiarities depend on the specific model one is considering; with the present chapter, we just wanted to highlight the problem and suggest a general method to deal with it (which was not explicitly encountered in the literature by us).

Before moving to a more physical part of the work, where we will provide an example of what has already been discussed, we just hint at some qualitative behaviours of singular systems — which will be studied in future works — and some extensions of the definitions above in the general d–dimensional case.

Indeed, beyond describing the local behaviour of orbits in a neighborhood of the singularity when $\boldsymbol{x}_s$ is degenerate (see Figure 3 (right)), and figure out some "natural condition" to impose at such a point in order not to lose uniqueness, one can also investigate what happens in the presence of some symmetries (e.g. when the singular curve is the axis of symmetry for the system), or when there are periodic orbits due to the singular curve.

Example 3 (Symmetric periodic orbits)**.** In the singular system

$$\begin{cases} \dot{x} = \dfrac{(x-1)(x-2)}{y} \\ \dot{y} = x - 3/2 \end{cases} \tag{2.32}$$

periodic orbits are generated by the presence of the singular curve.

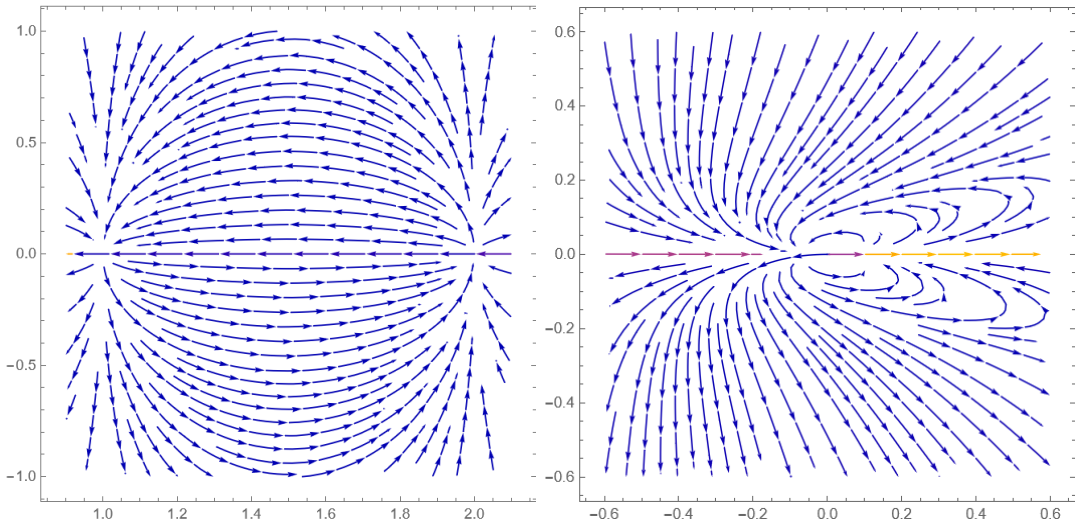

Figure 2.9: The phase space with (singular) periodic orbits of (2.32) on the left. On the right: the (R, H)–plane of the system generated by (3.27) and (3.28) with $f(R) = R + R^2$, around the degenerate singular point $(0, 0)$.

2.3 Autonomous singular systems

In the previous section we defined the associated regular system for an autonomous planar system of differential equations as (2.2). Nevertheless, the restriction to dimension two was made mainly with expository purposes, to settle the problem in relation to Arnold's geometric theory of differential equations. However, the main general ideas above can be stated for an autonomous system of generic finite dimension $d \in \mathbb{N}$, even though the dynamics in this case is much richer.

Therefore, we consider the singular autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}), \tag{2.33}$$

where $\boldsymbol{F} : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^d$ has components

$$F_i(\boldsymbol{x}) = \frac{\beta_i(\boldsymbol{x})}{\alpha_i(\boldsymbol{x})}, \quad \text{for} \quad i = 1, \dots, d, \tag{2.34}$$

and with $\alpha_1 \dots, \alpha_d, \beta_1, \dots, \beta_d \in C^1(\Omega)$.

Following the same main ideas of the latter section, it is possible to define the regular dynamical system associated with (2.33) as

$$\dot{\boldsymbol{x}} = \tilde{\boldsymbol{F}}(\boldsymbol{x}), \tag{2.35}$$

where the vector field components are

$$\tilde{F}_i(\boldsymbol{x}) = \beta_i(\boldsymbol{x}) \prod_{k \neq i} \alpha_k(\boldsymbol{x}), \quad \text{for} \quad i = 1, \dots, d. \tag{2.36}$$

One can easily check that the latter has the same $\partial x_i / \partial x_j$ as (2.33), and that the *singular hypersurface* is

$$\texttt{SHS} = \{\boldsymbol{x} \in \Omega \mid \alpha_1(\boldsymbol{x}) \dots \alpha_d(\boldsymbol{x}) = 0\}. \tag{2.37}$$

Remark 4. Notice that while classical solutions of the regular system (2.35) can be defined on the singular hypersurface, no Carathéodory solution of the singular system of equations (2.33) can be defined in an open interval of time in SHS as the vector field $\boldsymbol{F}$ is divergent Lebesgue–a.e. there.

As in the previous planar case, this provides an easy (dynamical) definition of a singular point:

Definition 2.3.1.

- A point $\boldsymbol{x}_s \in \Omega$ is a *proper singular point* of (2.33), if it is an equilibrium point of the system (2.35), but not of (2.33).
- A *nondegenerate (proper) singular point* of (2.33) is a proper singular point $\boldsymbol{x}_s \in \Omega$ such that $\det[D\tilde{\boldsymbol{F}}(\boldsymbol{x}_s)] \neq 0$.

In this general case, a variety of qualitatively different dynamical behaviours may take place — just think that the above "funnel points" would corresponds to sub–manifolds with dimension $d - 2$. However, reducing to the associated regular system (2.35) can still be useful while studying a particular model.

Chapter 3

Metric $f(R)$ theories of gravity

In this chapter, we present the basic motivations and features of the $f(R)$ theories of gravity, which follow from a straightforward generalization of the Einstein–Hilbert action when considering a general function f of the Ricci scalar curvature inside the Lagrangian.

3.1 Historical and experimental motivations

The more than a–century–old General Relativity (GR), introduced by Albert Einstein in 1915, is one of the most solid and confirmed physical theories ever conceived. After its publication, experimental results supporting its robustness were not long in coming: in 1919 Eddington provided the first observational verification of light bending during a solar eclipse. Despite Eddington's experiment, the fact that General Relativity is compatible with Newtonian gravity, and also accounts for the peculiar advance of the perihelion of the planet Mercury without relying on additional parameters ("fudge factors"), the theory did not gain widespread attention by theoretical physicists until the late 1950s, inaugurating the "golden age of General Relativity" between 1960 and 1975. This shift in popularity was further catalyzed by the discovery of highly energetic compact objects, leading to the subsequent development of the modern theory of gravitational collapse, black holes, and singularities.

However, recent data provide a new and unexpected picture of the universe, which is expanding at an increasing rate[1], contrary to what is expected by GR. Indeed, in the absence of cosmological constant and under the usual isotropy and homogeneity assumptions, from the Einstein field equations we can derive the *Friedmann equation*:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{3.1}$$

where a is the scale factor, G is the gravitational constant, and ρ and p denote respectively, the energy density and the pressure of the cosmological fluid which represents the matter. By (3.1) one deduces that if the Strong Energy Condition $\rho + 3p \geq 0$ holds, then there can be no acceleration of the expansion.

Thus, there is unquestionably a need for more thorough exploration to better elucidate the evolution of the universe's matter and energy content. One initial endeavor in this direction

[1]On the basis of these observations, the energy balance of the universe seems to be the following: 4% ordinary baryonic matter, 20% dark matter, and 76% dark energy [A$^+$06], [R$^+$04], where the term *dark matter* refers to an unknown form of matter, which possesses the clustering properties of ordinary matter but has not been observed directly as it does not appear to interact with electromagnetic fields. On the other hand, with the term *dark energy* we refer to an unknown form of energy that has not been directly detected, does not cluster like matter, and behaves like a cosmological constant affecting the universe on the largest scales. Indeed, due to its present abundance with respect to matter, it seems to drive an accelerating expansion of the universe.

is the "concordance model" or ΛCDM (Λ–Cold Dark Matter) that, however, is burdened by well-documented cosmological quandaries. One of these dilemmas pertains to the cosmological constant, as its observed value appears strikingly minute in comparison to the vacuum energy of matter fields [Car01]. Additionally, a natural question that arises is the following: given that the period when the energy density of the cosmological constant is comparable to that of the matter is incredibly brief relative to the lifespan of the universe, why does this synchronicity occur before our very eyes? This inquiry sums up what is usually referred to as *coincidence problem* and it is sometimes answered by invoking the *anthropic principle* [Gal81]. The latter, initially introduced by Dicke in 1957, posits that the range of conceivable observations about the universe is restricted by the requirement that these observations can only take place in a universe capable of fostering the development of intelligent life. However, this "solution" is rather controversial and not quite scientific, as one can argue that confuses the "cause" with the "effect"; therefore, physics and scientists are receptive to alternatives.

As the authors in [SF10] observe, a different approach to motivate such evidence, involves the potential modification of our theory of gravitation, covering the possibility that our current description of the gravitational interaction at relevant scales is not adequate. Indeed, it is conceivable that a more precise gravitational theory could provide explanations for phenomena like the ongoing accelerated expansion of the universe, without necessitating the introduction of enigmatic "dark" energy. This is reminiscent of the historical case of the precession of Mercury's orbit, which was initially attributed to the presence of an unseen planet orbiting within Mercury's trajectory and explained only later when GR was introduced.

Many attempts were pursued in this direction, from the non–viable first ones (cfr. [Wil18]) to the famous *scalar–tensor theory* [Ber68], to other more recent proposals — see for instance [DGP00], [Bek04]. In this chapter, we focus on another class of theories, namely the $f(R)$ theories of gravity, which allow for a more general dependence with respect to the Ricci scalar curvature in the gravitational Lagrangian.

In addition to the cosmological motivations above, there are other physical reasons to consider higher–order theories of gravity such as $f(R)$ theories. Reasons to question GR emerged already in the 1960s, when the great success found by high energy theories — like Quantum Field Theory (QFT) — led to the fact that GR is not renormalizable, and therefore not quantizable, being raised as an issue. Moreover, in 1962, Utiyama and De Witt proved that renormalization at one–loop requires higher order curvature terms to be included in the Einstein–Hilbert action [UD62]. In the same direction, other results were published, showing that higher order actions are indeed renormalizable (although not unitary) [Ste77], and that if quantum corrections or string theory are considered, the effective low energy gravitational action admits higher order curvature invariants [BD84], [BOS92], [Vil92].

Reasons, as those mentioned above, raised the interest in modified theories of gravity, in particular in higher–order theories of gravity, which are obtained by modifying the Einstein–Hilbert action by introducing higher–order curvature invariants — we refer to [Sch07] for a detailed review. Nevertheless, these terms were expected to be characterized by small coefficients (couplings) leading their relevance to be restricted to strong gravity regimes, i.e. close to the Planck scale, as it should happen close to black hole singularities, or in the early universe.

The $f(R)$ theories of gravity are defined by modifying the Einstein–Hilbert action:

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \tag{3.2}$$

where $\kappa := 8\pi G$, g is the determinant of the metric, and R is the Ricci scalar, by letting the Lagrangian have a more general dependence on R, i.e.

$$S := \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R). \tag{3.3}$$

There are several reasons to consider Lagrangians of such a form. On the one hand, under suitable conditions e.g. when f is analytic, the $f(R)$ actions are not too complicated to deal with — as compared to cases that consider other quantities (such as $R^{ab}R_{ab}$) as invariants. On the other side, they provide a description general enough to incorporate basic features of higher–order gravity and cosmology. Moreover, as exposed in [Woo07], there are good reasons to conjecture that only $f(R)$ theories, among higher–order gravity theories, escape the well–known Ostrogradski instability [MS20].

One could argue that high–energy modifications of the gravitational action, expected to be relevant only at the Planck scale, should not affect the late–time (with respect to the initial singularity) cosmological phenomenology. Even if this remark is correct, it is also dutiful to admit that gravity at a small scale is still well comprehended, and in any empirical approach there are some coefficients or constants that are relatively traceable, but still important (take for instance the mass of an electron with respect to classical mechanics). Following this spirit, there are already in the literature results claiming that quantities that affect late–time gravitational phenomena can be understood by a fundamental theory, such as string theory [NO03].

With the above motivations, in the next sections, we present the main features of the metric $f(R)$ theories of gravity, which, as said, provide non–trivial toy models to better investigate higher–order gravity and GR itself.

3.2 Field equations

As in the case of GR, there are two main approaches that one can adopt while varying the action to find the field equations. The former is the *standard metric variation*, which assumes that the curvature connection of the spacetime is the one associated with the metric of the manifold. The so–called *Palatini method*, instead, performs a variation with respect to both metric and the connection, which are assumed to be independent, together with the assumption of the independence of the matter action with respect to the connection.

The two different approaches lead to the same field equations for actions whose Lagrangian is a linear function of R, i.e. in the case of GR; this however is not true in general, leading one either to *metric $f(R)$ theories* or to *Palatini $f(R)$ theories*. For a review of the latter, and a comparison among several variational approaches we refer to [SF10], while for a standard treatment of the GR case a standard reference is [MTW73].

In the present work, we are focusing on *metric $f(R)$ theories*, since the Palatini counterpart still presents some open problems, concerning, for instance, the well–posedness of the Cauchy problem for the field equations (see [SF10]).

The choice of working with the Levi–Civita connection involves dealing with field equations that are PDE of the second order with respect to $R = g^{ab}R_{ab}$, and so of the fourth order with respect to the metric coefficients. We recall indeed that for a metric–compatible connection on a pseudo–Riemannian manifold $(\mathscr{M}, g)$ the coefficients are

$$\Gamma^a{}_{bc} = \left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\} := \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}), \tag{3.4}$$

where the Latin indices run from 1 to 4, and correspond to the space–time coordinates (t, x, y, z). In the case of the Levi–Civita connection, the Riemann curvature tensor reads

$$R_{abc}{}^d = \partial_b \Gamma^d{}_{ca} - \partial_a \Gamma^d{}_{cb} + \Gamma^d{}_{eb}\,\Gamma^e{}_{ca} - \Gamma^d{}_{ea}\,\Gamma^e{}_{cb}, \tag{3.5}$$

and so contains second order derivatives of the metric.

Considering a matter action S_M, together with the gravitational one in (3.3), leads to the following total action

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g_{ab},\psi), \tag{3.6}$$

where ψ denotes the matter fields.

In the context of GR, when we perform the variation of the action S_{EH} with respect to the metric g^{ab} and subsequently carry out an integration by parts within the space–time region $\mathscr{U} \subset \mathscr{M}$, we encounter certain surface terms. These quantities, when summed together, contribute to the total variation of a term which, in the case of GR, corresponds to the Gibbons–Hawking—York surface term. Incorporating this term into S_{EH}, we can define a gravitational action whose variation is free from boundary terms, i.e. has a well–defined functional derivative with respect to g^{ab}. However, because the surface terms that arise in the case of $f(R)$ gravity do not collectively amount to the variation of any quantity, this approach is not viable in this scenario. A way out is found by considering the presence of higher–order derivatives of the metric, which allows us to constrain additional degrees of freedom on the boundary by imposing $\delta g_{ab} = 0$ and $\delta R = 0$ on $\partial\mathscr{U}$. Although this process does not lead to a unique solution, it does yield the same field equations — for the details, we refer to [GCT10].

After those manipulations, one is left with the following field equations:

$$f_R(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - (\nabla_a\nabla_b - g_{ab}\Box)f_R(R) = \kappa T_{ab}, \tag{3.7}$$

where

$$T_{ab} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}}, \tag{3.8}$$

is the usual stress–energy–momentum tensor, and $\Box := \nabla^a\nabla_a$, with ∇_a denoting the covariant derivative with respect to the Levi–Civita connection. Notice that if $f(R) = -\Lambda + R/2$ as in GR, the field equations (3.7) reduce to the *Einstein field equations*:

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{3.9}$$

where $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab}$ is the Einstein tensor, and Λ is the cosmological constant.

The traces of (3.9) and (3.7) read respectively

$$R = 2\Lambda + \frac{\kappa}{2}T \tag{3.10}$$

and

$$f_R(R)R - 2f(R) + 3\Box f_R(R) = \kappa T, \tag{3.11}$$

where $T = g^{ab}T_{ab}$. Therefore, while in GR the scalar functions R and T are related algebraically, their dependence is differential in the general $f(R)$ case.

This fact already suggests that metric $f(R)$ theories are much richer, in the sense that they admit a larger class of solutions. For instance, the Jebsen–Birkhoff theorem [NGD14] — which states that the only spherically–symmetric vacuum solution of (3.9) is the Schwarzschild solution — is no longer valid for the general $f(R)$ equations (3.7).

Interestingly, notice that when investigating maximally symmetric solutions, for which the scalar curvature R =constant, in a vacuum ($T_{ab} = 0$), the equation trace (3.11) becomes an algebraic equation with respect to R:

$$f_R(R)R - 2f(R) = 0. \tag{3.12}$$

Now, if the latter has $R = 0$ as a solution, then this corresponds to the Minkowski spacetime as (3.7) reads $R_{ab} = 0$. The other possibility is that the solution of (3.12) is a non–zero constant C, which substituted into (3.7) leads to

$$R_{ab} = \frac{C}{4} g_{ab}, \tag{3.13}$$

i.e. the solution corresponds to the de Sitter or the anti–de Sitter spacetime.

3.3 Cosmological picture

As discussed, one of the main reasons supporting the study of $f(R)$ theories is the possibility to explain, without invoking obscure concepts like *dark energy*, the present observed acceleration in the cosmological expansion. In this section we briefly recall some basic features of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime and the main features (like the one above) of these models.

3.3.1 FLRW model for metric $f(R)$ theories

The high degree of isotropy measured in the cosmic microwave background is the main reason that drives cosmologists to consider the FLRW model. The latter refers to a solution of the field equations characterized by homogeneity and isotropy, practically obtained after considering the metric ansatz

$$g_{ab}\, dx^a dx^b = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right], \tag{3.14}$$

where (r, θ, ϕ) are comoving coordinates, and $a(t)$ is the scale factor of the universe — which is indeed expanding. The parameter K belongs to $\{-1, 0, 1\}$, and depending on its value, the universe is, respectively, hyperspherical, spatially flat, or hyperbolic. As already mentioned in the introduction, matter is usually included in the stress–energy–momentum tensor by defining

$$T^a{}_b := \mathrm{diag}(-\rho, p, p, p) \tag{3.15}$$

in terms of the energy density ρ and the pressure p of a perfect fluid which satisfies the continuity equation

$$a\dot{\rho} + 3\dot{a}(\rho + p) = 0. \tag{3.16}$$

Given that findings similar to the one established in by Ehlers, Geren, and Sachs [EGS68] — which asserts that when a set of timelike freely falling observers detect an isotropic radiation field, the spacetime must exhibit spatial homogeneity and isotropy, resembling a FLRW cosmological model — have been extended to metric $f(R)$ theories, as discussed in [RvETT96], it is suggested that the identification of the universe with a FLRW spacetime is reasonable, also in this case.

From now on, in analogy with the Einstein tensor, we denote the tensor on the left–hand side of the field equations (3.7) as

$$\mathcal{F}_{ab} := f_R(R) R_{ab} - \frac{1}{2} f(R) g_{ab} - (\nabla_a \nabla_b - g_{ab}\Box) f_R(R). \tag{3.17}$$

With this notation, after substituting (3.14) we find that the only non–zero components of (3.17) are

$$\mathcal{F}^0{}_0 = \frac{1}{\kappa} \left(-3 f_{RR} \dot{R} \frac{\dot{a}}{a} + 3 f_R \frac{\ddot{a}}{a} - \frac{f}{2} \right), \tag{3.18}$$

$$\mathcal{F}^1{}_1 = \mathcal{F}^2{}_2 = \mathcal{F}^3{}_3 = \frac{1}{\kappa}\left(-f_{RR}\ddot{R} - f_{RRR}\dot{R}^2 - 2f_{RR}\dot{R}\frac{\dot{a}}{a} + f_R\frac{\ddot{a}}{a} + f_R\frac{\dot{a}^2}{a^2} + 2f_R\frac{K}{a^2} - \frac{f}{2}\right). \quad (3.19)$$

Now, after the standard introduction of the Hubble parameter

$$H := \frac{\dot{a}}{a}, \quad (3.20)$$

the scalar curvature is explicitly written as

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) = 6\left(2H^2 + \dot{H} + \frac{K}{a^2}\right). \quad (3.21)$$

Notice that the following useful relations hold:

$$\dot{H} + H^2 = \frac{\ddot{a}}{a}, \quad \dot{H} = \frac{R}{6} - 2H^2 - \frac{K}{a^2}, \quad (3.22)$$

and that (3.15), (3.20) and (3.22) can be substituted in (3.18) and (3.19) to get

$$\mathcal{F}^0{}_0 = -3f_{RR}H\dot{R} - 3f_RH^2 + \frac{1}{2}(f_RR - f) - 3f_R\frac{K}{a^2} \quad (3.23)$$

$$\mathcal{F}^1{}_1 = -f_{RR}\ddot{R} - f_{RRR}\dot{R}^2 - 2f_{RR}\dot{R}H + f_R\dot{H} + 2f_RH^2 + 2f_R\frac{K}{a^2} - \frac{f}{2}. \quad (3.24)$$

After substituting (3.23) and (3.24) into the field equations (3.7), i.e. $\mathcal{F}^a{}_b = \kappa T^a{}_b$, the results can be rearranged in the so called *modified Friedmann equation*:

$$\mathcal{F}^1{}_1 - \mathcal{F}^0{}_0 = -f_{RR}\ddot{R} - f_{RRR}\dot{R}^2 + f_{RR}H\dot{R} - 2f_R\dot{H} + 2f_R\frac{K}{a^2} = \kappa(\rho + p), \quad (3.25)$$

and either by direct computation, or by means of the generalized Bianchi identity, one can show the relation

$$\frac{d\mathcal{F}^0{}_0}{dt} = 3H(\mathcal{F}^1{}_1 - \mathcal{F}^0{}_0), \quad (3.26)$$

which is consistent with the continuity equation (3.16).

The independent differential equations which govern the evolution are (3.20), (3.21), (3.23), and (3.16). Therefore if we express the relation between matter density and pressure as $p = p(\rho)$, it is possible to exhibit a set of first order differential equations for a, H, R, and ρ:

$$\dot{R} = \frac{1}{3f_{RR}(R)H}\left[\kappa\rho + \frac{f_R(R)R - f(R)}{2} - 3f_R(R)H^2 - f_R(R)\frac{K}{a^2}\right], \quad (3.27)$$

$$\dot{H} = \frac{R}{6} - 2H^2 - \frac{K}{a^2}, \quad (3.28)$$

$$\dot{\rho} = -3H(\rho + p), \quad (3.29)$$

$$\dot{a} = Ha. \quad (3.30)$$

Notice that, although the field equations are of fourth order in $a(t)$, in the spatially–flat case the equations decouple, and the scale factor only appears in the latter two. Since the Hubble parameter is a cosmological observable and the equations (3.27) and (3.28) are of second order with respect to H, it is convenient and common to adopt the latter as a first dynamical variable.

Except for some particular cases, the above equations are not solvable analytically, and usually one either employs numerical methods to approximate solutions or deduce the general behaviour of solutions by studying the phase space. For instance, in a spatially flat FLRW universe (as the equations (3.27) and (3.28) decouple from the others) a common and handy choice

of phase space variables is (R, H), and so the latter can be viewed as a two–dimensional sub–manifold of the three–dimensional one $(\dot{R}, R, H)$ with de Sitter spaces as fixed points [dSF07].

Furthermore, notice that the spatially flat, two–dimensional dynamical system described by (3.27) and (3.28) has the form of the system (2.19). In particular, it will be interesting to study the properties of singular points, which lie on the singular line $H = 0$.

Remark 5. Note that in the spatially flat case, and in the absence of matter, the system can be solved by decoupling the planar system in R and H. Actually, the choice $K = 0$ is quite common since it simplifies computations a lot, and sometimes it is claimed that this choice is also supported by the data. However, the latter fact is not entirely true, in fact, even though recent data indicate an estimate of the value of the density parameter

$$\Omega_K = -\frac{K}{a^2 H^2} \tag{3.31}$$

close to zero — considering the current rate of expansion of the universe $\dot{a}(t)$ which is very high.

3.3.2 Accelerating expansion

Aware of the above observation, we continue our dissertation with the assumption of spatial flatness, $K = 0$, since as far as "late–time" cosmology is concerned, this choice does not lead to great loss of accuracy.

As discussed above, an argument supporting metric $f(R)$ theories is that they can lead to accelerated expansion without invoking the presence of "dark energy". A way to show this is to define an effective energy density and pressure of the geometry as follows.

Firstly, the substitution of (3.23) and (3.24) into the field equations $\mathcal{F}^a{}_b = \kappa T^a{}_b$ gives

$$H^2 = \frac{1}{3f_R}\left[\kappa\rho + \frac{Rf_R - f}{2} - 3f_{RR}H\dot{R}\right], \tag{3.32}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f_R}\left[\kappa p + f_{RRR}\dot{R}^2 + 2f_{RR}H\dot{R} + f_{RR}\ddot{R} - \frac{Rf_R - f}{2}\right]. \tag{3.33}$$

Then, we introduce

$$\rho_{eff} := \frac{Rf_R - f}{2\kappa\, f_R} - \frac{3f_{RR}H\dot{R}}{\kappa\, f_R}, \tag{3.34}$$

$$p_{eff} := \frac{f_{RRR}\dot{R}^2 + 2f_{RR}H\dot{R} + f_{RR}\ddot{R} - (Rf_R - f)/2}{\kappa\, f_R}, \tag{3.35}$$

and observe, by taking the limit $\rho \to 0$ in (3.32), that ρ_{eff} is non–negative in a spatially flat FLRW spacetime. With this rearrangement — *in vacuo*, i.e. $\rho = p = 0$ — the equations (3.32), and (3.33) can be rewritten in the same form as the Friedmann equations (cfr. (3.1)):

$$H^2 = \frac{\kappa}{3}\rho_{eff} \tag{3.36}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho_{eff} + 3p_{eff}). \tag{3.37}$$

Therefore, in an empty universe, we can interpret the extra curvature corrections — due to the choice of an $f(R)$ theory — as an effective fluid which takes the role of the "dark energy/matter" to motivate the observed cosmic acceleration. One can also define the *state parameter*

$$w_{eff} := \frac{p_{eff}}{\rho_{eff}} = \frac{\dot{R}^2 f_{RRR} + 2H\dot{R}f_{RR} + \ddot{R}f_{RR} + (f - Rf_R)/2}{(Rf_R - f)/2 - 3H\dot{R}f_{RR}}, \tag{3.38}$$

which has the same sign of its numerator, as the denominator is always positive because of equation (3.32). Hence, in a general context, in order for a metric $f(R)$ theory to replicate the de Sitter equation of state, where $w_{eff} = -1$, it has to satisfy the condition:

$$\frac{f_{RRR}}{f_{RR}} = \frac{\dot{R}H - \ddot{R}}{\dot{R}^2}. \tag{3.39}$$

Example 4. If one considers functions $f(R) \propto R^n$ and assumes in order to have a time–independent w_{eff}, the scale factor to be of the form $a(t) = a_0 \, (t/t_0)^{\alpha}$, one finds

$$w_{eff} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}, \quad \text{for} \quad n \neq 1, \tag{3.40}$$

and

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{3.41}$$

Hence, in order to be $w_{eff} = -1$, it must hold $n = 2$ and $\alpha = \infty$, in agreement with the fact that quadratic corrections to S_{EH} were considered in Starobinasky inflation [Sta80].

3.3.3 Cosmological eras

We showed how a modified $f(R)$ Lagrangian could possibly explain a cosmic speed–up at late times. Nevertheless, in order to propose a cosmological model that aligns with the observed data, it is necessary for the theory to exhibit a sequence of epochs akin to the following one, which we present without the details that can be found, for instance, in [PSTK94], [CL03].

1) **An early inflation**: a period of exponential expansion of the early universe after the Big Bang, with an estimated duration between 10^{-36} to 10^{-32} seconds, and when the temperature drops from 10^{27} to $10^{22} K$.

2) **A radiation era**: when the energy density of the electromagnetic radiation is greater than the energy density of matter. This is also usually divided into other three periods:

   i) **Quark Epoch**: characterized by the creation of neutrons and protons for a time–lap of about 10^{-4} seconds, and with a further decrease in temperature down to 10^{13} K.

   ii) **Lepton Epoch**: during which leptons (electrons, neutrinos, and other light particles), are produced — still in pairs due to their small mass. After about 10^2 seconds, the temperature is also estimated to have dropped to about $10^9 K$.

   iii) **Nuclear Epoch**: within this estimated 15–minute time interval, the conditions of pressure and temperature in the universe facilitate the fusion of protons and electrons into nuclei.

3) **A matter era**: when the energy density of electromagnetic radiation is lower than the energy density of matter, and we can further distinguish a

   i) **Atomic Epoch**: covering the period between 50.000 to 200.000 years after the Big Bang, and characterized by the decoupling of the electromagnetic radiation, the formation of atoms, and the appearance of the cosmic microwave background.

   ii) **Galactic Epoch**: when the large–scale structure of most of the galaxies was formed, between 200 million and 3 billion years after the Big Bang.

   iii) **Stellar Epoch**: with the formation of the lower scale structure of galaxies: stars and planets.

4) **The present accelerated epoch**, and

5) **A future era**.

As in the above–accredited version of the cosmological *historia*, the *matter era* should last long enough[2] to allow the formation of the structures (galaxies and stars), while the *future era* is usually modeled as a de Sitter attractor solution, or truncated via the introduction of a Big Rip singularity [S+17].

Moreover, another requirement is to have a smooth transition between two consecutive eras, and the latter happens to be a much stricter requirement (compared to one mentioned above) for the metric $f(R)$ theories. In fact, in many examples — including the form $f(R) = R - \mu^{2(n+1)}/R^n$, $n > 0$ — the problem of a smooth transition has been studied and claimed to be, at least, problematic [AGPT07], [NO06].

In principle, it is possible to find choices of the function $f(R)$ that satisfy the above requirements and match the astrophysical data. Indeed, since the Hubble parameter — more precisely viewed as a function $H(z)$ of the redshift parameter z — can be reconstructed by experimental measurements, one can determine an appropriate $f(R(z))$ by (numerically) integrating the third order linear differential equations (3.23) and (3.24). This procedure, studied in [CCT05], provides a class of $f(R)$ functions which, in theory, best reproduces the observed cosmological dynamics. However, this is not sufficient to cover the system's degrees of freedom; in other words, the Hubble parameter $H(t)$ alone is not enough to uniquely reconstruct $f(R)$. The additional information usually comes from cosmological density perturbations, which are the key to understanding the evolution of the large–scale structure of the universe.

3.4 Dolgov–Kawasaki instability

Other relevant constraints on the form of the function $f(R)$, come from some stability remarks that rely on how the theory behaves in regimes that are well approximated by GR. For instance, it is believed that a stable ground state of the universe should be highly symmetric, like de Sitter or Minkowski spacetime. Dolgov and Kawasaki [DK03] discussed the instability in the particular model where

$$f(R) = R - \mu^4/R, \tag{3.42}$$

and the analysis was then generalized for arbitrary metric $f(R)$ theories by Faraoni [Far06], whose main ideas are reported below.

To highlight the deviation from GR we express

$$f(R) = R + \epsilon\varphi(R), \tag{3.43}$$

where ε is a small parameter with dimension of mass squared, and φ is adjusted to be dimensionless — e.g. for the equation (3.42) one has that $\varepsilon = \mu^2$, $\varphi = -\mu^2/R$, and $\mu \approx 10^{-33}eV$. With these notations, and with the assumption that $\varphi_{RR} \neq 0$ in order to deviate from GR[3], the trace equation (3.11) reads

$$\Box R = \frac{\varphi_{RRR}}{\varphi_{RR}}\nabla^a R\nabla_a R + \frac{\varepsilon\varphi_R - 1}{3\varepsilon\varphi_{RR}}R = \frac{\kappa T}{3\varepsilon\varphi_{RR}} + \frac{2\varphi}{3\varphi_{RR}}. \tag{3.44}$$

[2]For instance, the $f(R) = R - \mu^4/R$ model fits this condition [EHC08].

[3]Isolated zeros of φ_{RR} where the theory is "instantaneously GR" are usually not considered in the literature. As we will observe in the next chapter, in this case, the singular points of the spatially–symmetric system in the (R, H)–plane are non–regular, and their analysis is more complicated.

We consider a small region of spacetime characterized by a weak–field regime, and we locally approximate the metric and scalar curvature as

$$g_{ab} = \eta_{ab} + h_{ab}, \quad R = -\kappa T + R_1, \tag{3.45}$$

where η_{ab} denotes the Minkowski metric and $|R_1/\kappa T| \ll 1$. Substituting in (3.44) gives

$$\begin{aligned} \ddot{R}_1 - \boldsymbol{\nabla}^2 R_1 - \frac{2\kappa\varphi_{RRR}}{\varphi_{RR}}(\langle \boldsymbol{\nabla} T, \boldsymbol{\nabla} R_1\rangle - \dot{T}\dot{R}_1) + \frac{1}{3\varphi_{RR}}\left(\frac{1}{\varepsilon} - \varphi_R\right) R_1 = \\ = \kappa\ddot{T} - \kappa\boldsymbol{\nabla}^2 T - \frac{(\kappa T\varphi_R + 2\varphi)}{3\varphi_{RR}}, \end{aligned} \tag{3.46}$$

where $\boldsymbol{\nabla}$, and $\boldsymbol{\nabla}^2$ are respectively the three–dimensional gradient and Laplacian, and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$. One can now observe that, as the coefficient of R_1 is very large in modulus for small values of ε which reproduce the correct cosmological dynamics, the scalar mode R_1 of the $f(R)$ theory is stable if $\varphi_{RR} = f_{RR} > 0$ and unstable if $\varphi_{RR} = f_{RR} < 0$. Indeed, in this regime, the leading terms of (3.46) are in the form of a Klein–Gordon field equation

$$\ddot{\psi} - \boldsymbol{\nabla}^2\psi + m^2\psi = 0, \quad m^2 \in \mathbb{R}, \tag{3.47}$$

whose solutions can be expressed by Fourier transform as a combination of exponential fundamental modes whose time–component is $e^{\pm imt}$, and so stable, if $\varphi_{RR} > 0$, while it is $e^{\pm mt}$, i.e. unstable, if $\varphi_{RR} < 0$.

The expected time scale for the instability to take place is of the order of the inverse effective mass (in the case of (3.42) it is approximately 10^{-26} seconds).

Other instabilities can be investigated, like the Gauge–invariant instability for the de Sitter spacetime (cfr. [SF10]). Nevertheless, we recalled the Dolgov–Kawasaki instability, also known as "Ricci scalar instability", to motivate the choice $f_{RR} > 0$ in the following discussion.

Chapter 4

Singular points in $f(R)$ models

In this chapter, we analyze the (R, H)–phase plane of the planar singular system of the empty FLRW spatially flat model determined by (3.27) and (3.28). Notice that the latter, which we recall to be

$$\dot{R} = \frac{1}{3f_{RR}(R)H}\left[\frac{f_R(R)R - f(R)}{2} - 3f_R(R)H^2\right] =: \rho(R, H), \tag{4.1}$$

$$\dot{H} = \frac{R}{6} - 2H^2 =: \eta(R, H), \tag{4.2}$$

has the same form as the singular systems (2.19) studied in Chapter 2, letting

$$\alpha(R, H) = 6f_{RR}(R)H, \tag{4.3}$$

$$\beta(R, H) = f_R(R)(R - 6H^2) - f(R), \tag{4.4}$$

$$\delta(R, H) = \frac{R}{6} - 2H^2. \tag{4.5}$$

From now on, in order to apply the previous results, we will assume that $f \in C^3(I)$ in an open interval $I \subseteq \mathbb{R}$, and that $f_{RR}(R) > 0$ in order to avoid the Dolgov–Kawasaki instability[1].

Example 5. Later on, we will consider for instance Lagrangians of the form

$$f(R) = a + b\,R + c\,R^2, \tag{4.7}$$

with a, b, c real, non–zero parameters, $c > 0$; which therefore satisfy the assumptions above.

Before addressing the analysis of singular points, we first deal with equilibria and their stability.

4.1 Stationary points

The *equilibria* or *stationary points* of the system [SA] are the points (R_e, H_e) in which the vector field vanishes, namely

$$\begin{cases} f_R(R_e)R_e = 2f(R_e) \\ R_e = 12H_e^2. \end{cases} \tag{4.8}$$

[1]Notice that, if we restrict the vector field domain to

$$\Omega = \{(R, H) \mid R > 12H^2, \quad \text{and} \quad R \in I\}, \tag{4.6}$$

all the assumption made above Definition (2.1.1) are satisfied.

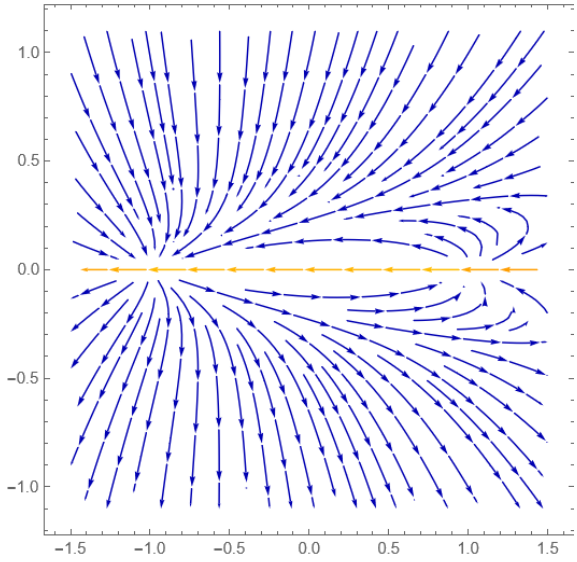

Figure 4.1: Phase space of the equations (4.1) and (4.2) with f as in (4.7) (with $a = b = c = 1$ in the plot). Notice that a Carathéodory solution which crosses the singularity is a *bouncing cosmological solution*, i.e. it represents a universe that is contracting till it reaches the singularity after which it starts to expand (or vice–versa).

It follows that the equilibria of the system lie on the parabola $R = 12H^2$. Moreover, notice that if $f(R) = \alpha R^2$, with $\alpha \in \mathbb{R}$, the first equation above is always satisfied and the whole curve is composed of equilibrium points.

We can use the remark above to introduce a geometric criterion to find roots of $f_R R - 2f$, hence equilibria. Indeed, let us fix a point $\tilde{R} \in I$; then there is one of the one–parameter family of parabolas $g^\alpha(R) := \alpha R^2$ which intersects the graph of f at $(\tilde{R}, f(\tilde{R}))$, in other terms $g^{\alpha(\tilde{R})}(\tilde{R}) = f(\tilde{R})$. Since the parabola satisfies the first of (4.8) identically, the condition that $f_R(\tilde{R})\tilde{R} - 2f(\tilde{R}) = 0$ is verified if and only if

$$f_R(\tilde{R}) = (g^{\alpha(\tilde{R})})_R(\tilde{R}). \tag{4.9}$$

We conclude that f has a point such that the first in (4.8) holds, if and only if a quadratic curve with the maximum/minimum at the origin is tangent to f.

It is therefore possible to find the equilibria of the system by considering the one–parameter family of parabolas $g^\alpha(R) := \alpha R^2$ and their intersection with f in the $(R, f(R))$–plane.

4.1.1 Stability of the stationary points

To investigate the stability of the equilibrium points, we study the eigenvalues of the Jacobian matrix $\mathbb{J} := D\boldsymbol{F}(R, H)$ of the vector field. The latter reads

$$D\boldsymbol{F}(R,H) = \begin{pmatrix} -\frac{f_{RRR}(R)}{f_{RR}(R)}\rho(R,H) + \frac{1}{6H}(R - 6H^2) & -\frac{\rho(R,H)}{H} - 2\frac{f_R(R)}{f_{RR}(R)} \\ 1/6 & -4H \end{pmatrix}, \tag{4.10}$$

where $\boldsymbol{F}(R, H) := (\rho(R, H), \eta(R, H))$. Recalling that the characteristic polynomial of $\mathbb{J}$ is

$$p_{\mathbb{J}}(\lambda) = \det(\mathbb{J} - \lambda\mathbb{I}) = \lambda^2 - (\operatorname{tr}\mathbb{J})\,\lambda + \det\mathbb{J}, \tag{4.11}$$

we have

$$\det \mathbb{J} = 4H\frac{f_{RRR}(R)}{f_{RR}(R)}\rho(R,H) + \frac{\rho(R,H)}{6H} - \frac{2}{3}(R-6H^2) + \frac{1}{3}\frac{f_R(R)}{f_{RR}(R)}, \tag{4.12}$$

and

$$\operatorname{tr} \mathbb{J} = -\frac{f_{RRR}(R)}{f_{RR}(R)}\rho(R,H) + \frac{R}{6H} - 5H. \tag{4.13}$$

At the stationary points the condition $\dot{H} = 0$ reads

$$H_e = \pm\sqrt{\frac{R_e}{12}}. \tag{4.14}$$

Because of the phase plane symmetry (the stability properties are inverted by $H \mapsto -H$) we can restrict ourselves in studying the positive square root, which substituted gives

$$\det \mathbb{J} = 2\sqrt{\frac{R_e}{3}}\left(\frac{f_{RRR}(R_e)}{f_{RR}(R_e)} + \frac{1}{2R_e}\right)\rho(R_e,H_e) - \frac{R_e}{3} + \frac{1}{3}\frac{f_R(R_e)}{f_{RR}(R_e)} = \frac{1}{3}\left(\frac{f_R(R_e)}{f_{RR}(R_e)} - R_e\right), \tag{4.15}$$

$$\operatorname{tr} \mathbb{J} = -\frac{f_{RRR}(R_e)}{f_{RR}(R_e)}\rho(R_e,H_e) - \sqrt{\frac{3R_e}{4}} = -\sqrt{\frac{3R_e}{4}}, \tag{4.16}$$

since $\rho(R_e, H_e) = 0$. The discriminant of the characteristic polynomial (4.11) in the equilibria reads

$$\Delta = (\operatorname{tr} \mathbb{J})^2 - 4\det \mathbb{J} = \frac{7}{12}R_e - \frac{4}{3}\frac{f_R(R_e)}{f_{RR}(R_e)}. \tag{4.17}$$

As $R_e \geq 0$, we deduce that if $f_R(R_e)/f_{RR}(R_e) < 0$ holds, then $\det \mathbb{J} < 0$ and the equilibrium is a saddle point. Conversely, if $f_R(R_e)/f_{RR}(R_e) \geq 0$ there are four possibilities:

1) $\det \mathbb{J} < 0$, (saddle point),

2) $\det \mathbb{J} = 0$, (higher order terms need to be considered for the stability analysis),

3) $\det \mathbb{J} > 0$ and $\Delta > 0$, (attractor point),

4) $\det \mathbb{J} > 0$, and $\Delta < 0$, (repeller point).

It is again possible to give a graphical interpretation of the above cases. Since

$$R_e = g_R^{\alpha(R_e)}(R_e)/g_{RR}^{\alpha(R_e)}(R_e) \tag{4.18}$$

the first and second cases correspond respectively to

$$\frac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} < 1 \quad \text{and} \quad f_{RR}(R_e) = g_{RR}^{\alpha(R_e)}(R_e), \tag{4.19}$$

while the last two cases correspond to

$$\frac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} > 1 \quad \text{and} \quad \frac{f_R(R_e)}{f_{RR}(R_e)} < \frac{7}{16}R_e, \tag{4.20}$$

and

$$\frac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} > 1 \quad \text{and} \quad \frac{f_R(R_e)}{f_{RR}(R_e)} > \frac{7}{16}R_e. \tag{4.21}$$

Note that if a stationary point turns out to be a saddle one, since $\dot{H} > 0$ in $R > 12H^2$ and it is negative outside, then one of its separatrices is contained in the parabola. We summarize the classification above in the following table:

Condition	**Equilibrium**
$f_R(R_e)/f_{RR}(R_e) < 0$	Saddle point
$f_R(R_e)/f_{RR}(R_e) \geq 0$ and $\dfrac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} < 1$	Saddle point
$f_R(R_e)/f_{RR}(R_e) \geq 0$ and $f_{RR}(R_e) = g_{RR}^{\alpha(R_e)}(R_e)$	Unknown
$f_R(R_e)/f_{RR}(R_e) \geq 0$, $\dfrac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} > 1$ and $\dfrac{f_R(R_e)}{f_{RR}(R_e)} < \dfrac{7}{16}R_e$	Attractor point
$f_R(R_e)/f_{RR}(R_e) \geq 0$, $\dfrac{f_{RR}(R_e)}{g_{RR}^{\alpha(R_e)}(R_e)} > 1$ and $\dfrac{f_R(R_e)}{f_{RR}(R_e)} > \dfrac{7}{16}R_e$	Repeller point

Example 6. In the case of Lagrangians of the form $f(R) = a + bR + cR^2$, the equilibria are

$$(R_e, H_e) = \left(\frac{-2a}{b}, \pm\sqrt{\frac{-a}{6b}}\right), \tag{4.22}$$

so they exist if and only if $a \cdot b < 0$. If this is the case, by (4.15) and (4.16) one respectively finds that

$$\det \mathbb{J} = \frac{b}{6c}, \quad \text{and} \quad \Delta = (\operatorname{tr}\mathbb{J})^2 - 4\det\mathbb{J} = \frac{9ac - 4b^2}{6cb}, \tag{4.23}$$

So if $b < 0$ they are saddle points, while if $b > 0$, they are attractors or repellers respectively if $9ac > 4b^2$ or $9ac < 4b^2$.

4.2 Singular points

We can now finally study the dynamics near the singular points of the system determined by (4.1) and (4.2). As $f_{RR} > 0$, the singular points lie on the singular curve $\alpha(R, H) = H = 0$, and they are individuated by the additional equation

$$\beta(R, H) = f_R(R)(R - 12H^2) - f(R) = 0. \tag{4.24}$$

We first notice that, under our assumptions, there exist at most two singular points.

Lemma 1. *If $f \in C^3(I)$, and $f_{RR} > 0$ in $I \subseteq \mathbb{R}$, then the equation*

$$f_R(R)R = f(R) \tag{4.25}$$

admits at most two solutions in I.

Proof. Observe that

$$\frac{d}{dR}\Big(f_R(R)R - f(R)\Big) = f_{RR}(R)R \geq 0, \tag{4.26}$$

which is continuous by assumption and strictly positive away from zero. Therefore, if the function $g(R) := f_R(R)R - f(R)$ has a root in zero, then $R_s = 0$ is the only solution to (4.25). Otherwise, there can be at most two solutions in I (which in this case have opposite sign as $g_R(0) = 0$). □

Example 7. In our example $f(R) = a + bR + cR^2$ with $c > 0$, the singular points satisfy

$$R_s = \pm\sqrt{\frac{a}{c}}, \quad \text{and} \quad H_s = 0, \tag{4.27}$$

and so they exist if and only if $a > 0$.

The last assumption that we make, in order not to have a triple intersection among $\alpha = 0$, $\beta = 0$, and $\delta = 0$, is that $f(0) \neq 0$. In particular, from what is discussed in the above Proof, if one restricts Ω to positive/negative curvatures then there is a unique singular point.

Remark 6 (Local shape of the integral curves)**.** We can find an approximation of the shape of integral curves in a neighborhood of a singular point $(R_s, 0)$ by considering

$$\begin{aligned}\frac{dR}{dH}(H) = \frac{\rho(R,H)}{\eta(R,H)} &= \frac{f_R(R)R - f(R) - 6f_R(R)H^2}{f_{RR}(R)H(R-12H^2)} \\ &= \frac{f_R(R)R - f(R)}{f_{RR}(R)R\cdot H - 12f_{RR}(R)H^2} - \frac{6f_R(R)H}{f_{RR}(R)R - 12f_{RR}(R)H^2},\end{aligned} \tag{4.28}$$

and neglecting the higher–order terms as

$$\frac{dH}{H} \approx \frac{f_{RR}(R)R}{f_R(R)R - f(R)}dR = \frac{d}{dR}\Big(\log(f_R(R)R - f(R))\Big)dR.$$

After integrating, we have that the local shape of the integral curves is

$$C_1 \cdot |H| \approx |f_R(R)R - f(R)|, \quad C_1 \in \mathbb{R} \tag{4.29}$$

and[2], if $C_1 \neq 0$, also that $\frac{dH}{dR}(R_s) \approx \frac{f_{RR}(R_s)R_s}{C_1}$. If $C_1 = 0$ the integral curves in $(R_s, 0)$ are parallel to the H–axis. In practical terms, the choice of the integration constant C_1 uniquely determines a particular incoming solution, the one with incoming direction $f_{RR}(R_s)R_s/C_1$ in phase space. Recalling what was discussed in Chapter 2, this fact already hints that the singular points are nondegenerate and incident funnel points.

4.2.1 Regular associated system

Following the procedure presented in Chapter 2, we study the local behaviour of solutions around singular points by writing the regular associated system to

$$\begin{cases} \dot{R} &= \dfrac{1}{f_{RR}(R)H}\left[\dfrac{1}{6}\Big(f_R(R)R - f(R)\Big) - f_R(R)H^2\right], \\ \dot{H} &= R/6 - 2H^2, \end{cases} \tag{4.30}$$

which is

$$\begin{cases} \dot{R} = f_R(R)(R - 6H^2) - f(R) =: \tilde{\rho}(R,H) \\ \dot{H} = H\cdot f_{RR}(R)(R-12H^2) =: \tilde{\eta}(R,H). \end{cases} \tag{4.31}$$

Notice that the latter ODEs, under suitable smoothness conditions over f (like ours[3]), do define a dynamical system. Therefore, we can employ the basic tools of the theory of smooth dynamical systems, in particular, the Poincaré–Bendixon theorem. Moreover, by applying Remark 8, we deduce that such a system is equivariant with respect to the reflection over the R–axis.

[2]Assuming to "include" in C_1 the sign arbitrariness due to the presence of the absolute value.

[3]Since we are assuming $f(R) \in C^3(R_l, R_r)$, the vector field of (4.31) is $C^1(\Omega)$, and so locally Lipschitz–continuous in Ω.

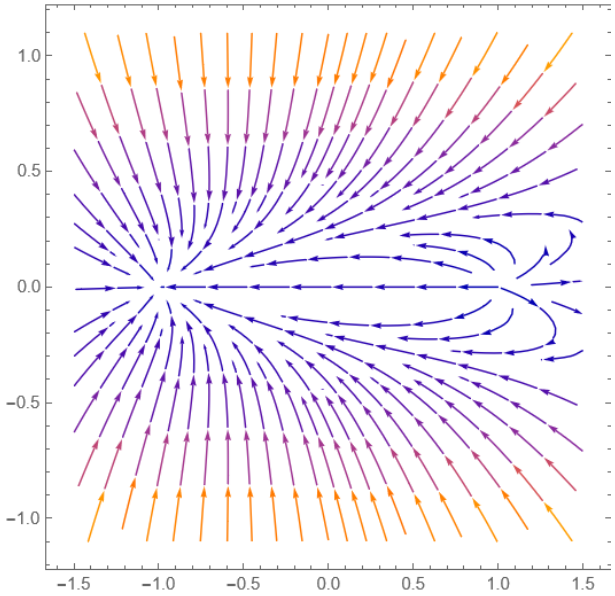

Figure 4.2: Vector field of (4.31) with $f(R) := 1 + R + R^2$. Notice that orbits are sent by the equivariance (reflection) into other orbits preserving the time direction.

As expected, the singular points of our system (4.30) — for which $H = 0$ and (4.25) hold — correspond to stationary points of the corresponding equivariant system. In these points, we can therefore perform a linearization procedure in order to study their stability. By letting $\tilde{\boldsymbol{F}} := (\tilde{\rho}, \tilde{\eta})$ we have that the Jacobian of (4.31) is

$$D\tilde{\boldsymbol{F}}(R,H) = \begin{pmatrix} f_{RR}(R)(R-6H^2) & -12H \cdot f_R(R) \\ H \cdot (f_{RRR}(R)(R-12H^2) + f_{RR}(R)) & f_{RR}(R)(R-36H^2) \end{pmatrix}, \tag{4.32}$$

that evaluated in the equilibria of the form $(R_s, 0)$ reads[4]

$$D\tilde{\boldsymbol{F}}(R_s,0) = f_{RR}(R_s)R_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.33}$$

therefore, the associated equilibrium point is a stable (resp. unstable) node if $f_{RR}(R_s)R_s < 0$ (resp. $f_{RR}(R_s)R_s > 0$) with the same real eigenvalues with geometric multiplicity 2.

Example 8. In the case $f(R) = a + bR + cR^2$ the Jacobian matrix is

$$D\tilde{\boldsymbol{F}}(\pm\sqrt{a/c}, 0) = \pm 2\sqrt{ac} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.34}$$

Now, by Theorem 1 and the remarks below it follows that the singular points (if present) are actually nondegenerate funnel points and the Carathéodory solutions reach them in finite time by all possible directions in the phase (R, H)–plane.

As we hinted in Chapter 2, since singular points may correspond to equilibria of the regular associated system possessing a basin of attraction, their importance, from a dynamical perspective, is not only local. For instance, if in our case the function f is defined over a bounded interval I, as it happens in *bounded curvature models* [FZ21], the following result holds.

[4]Notice that the quantity $f_{RR}(R_s)R_s$ appears again.

Theorem 2. *Assume that $f \in C^3(I)$, and that $f_{RR} > 0$ in the bounded open domain $I \subseteq \mathbb{R}$. If the system*

$$\begin{cases} \dot{R} &= \dfrac{1}{f_{RR}(R)H}\left[\dfrac{1}{6}\Big(f_R(R)R - f(R)\Big) - f_R(R)H^2\right], \\ \dot{H} &= R/6 - 2H^2, \end{cases} \tag{4.35}$$

contains a singular point $(R_s, 0)$ in

$$\Omega := \{(R, H) \mid R > 12H^2 \quad \textit{and} \quad R \in I\}, \tag{4.36}$$

then all the orbits in Ω contain the nondegenerate singular point $(R_s, 0)$.

Proof. Notice that $R, R_s > 0$ and that the singular point $(R_s, 0)$ is unique in Ω. As the second component of the vector field is always positive in Ω, i.e. $\dot{H} = R/6 - 2H^2 > 0$, a Carathéodory solution with initial values in $\Omega_- := \Omega \cap \{H < 0\}$ reaches the singular line in a finite time, and therefore has to cross it at the singular point $(R_s, 0)$ by Proposition 1. By recalling that the system is reversible with respect to the R–axis, we have the thesis also for solutions generated in Ω_+ with time reversed. □

Example 9 (Bounded hemi–circular model)**.** The previous result applies if we consider

$$f(R) := -\sqrt{(R - R_r)(R_l - R)}, \tag{4.37}$$

with $0 < R_r < R_l$ defined in the interval $I = (R_r, R_l)$.

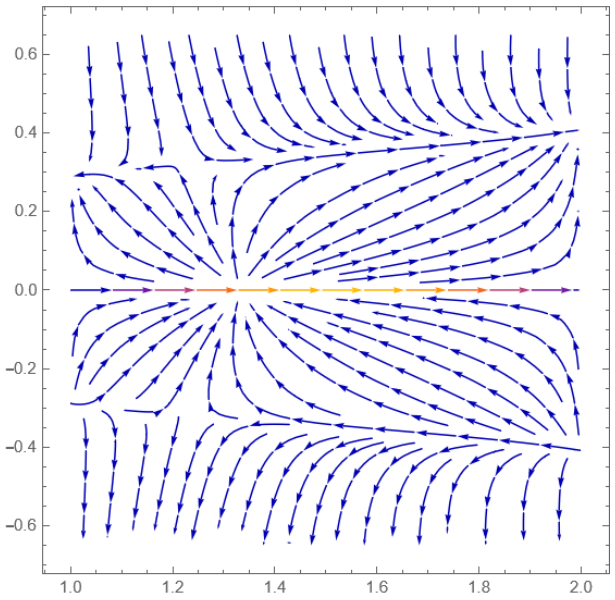

Figure 4.3: Vector field of the singular planar system with f as in (4.37). The singular point in Ω is $(R_s, H_s) = (4/3, 0)$.

4.2.2 Additional condition

We mentioned earlier that it is interesting to consider some additional conditions to continue a solution past a non–generic point uniquely.

A first attempt was suggested already in Remark 6: choose the solution with the same direction (to be intended as the slope in phase space) as the one of the incoming solution. In

other terms, if the incoming solution has slope $dH/dR(R_s) = \alpha$ we can choose the integration parameter in (4.29) as

$$C_1 := f_{RR}(R_s)R_s/\alpha, \tag{4.38}$$

and, since the singular point is a funnel point of the first type, we recover the uniqueness of solutions with initial data in an open set $V_* = V - \{H = 0\}$, where V is an open neighborhood of $(R_s, 0)$.

Remark 7. Notice that this choice is equivalent to extending the right–hand side of (4.30) imposing $\rho(R_s, 0) = R_s/(6\alpha)$ in $(R_s, 0)$. Indeed, one can see that

$$\rho(R_s, 0) := \frac{dR}{dH}(0) \cdot \dot{H}(\bar{t}) = \left(\frac{dH}{dR}(R_s)\right)^{-1} \dot{H}(\bar{t}) = \frac{1}{\alpha} \cdot \frac{R_s}{6} = \frac{R_s}{6\alpha}, \tag{4.39}$$

or, equivalently, substitute (4.38) inside of (4.1)[5].

Interestingly, the last quantity in (4.39) coincides with the limit of $\rho(R, H)$ as (R, H) tends to $(R_s, 0)$ with direction $dH/dR = \alpha$, as one can easily see by Taylor–expanding ρ in powers of $\Delta R := (R - R_s)$, see Appendix C.1.

Summarizing, with respect to a certain orbit that approaches $(R_s, 0)$ with direction $dH/dR = \alpha$ in phase space, $\rho(R, H)$ has limit $\frac{R_s}{6\alpha}$. Therefore choosing such a value to extend $\rho(R, H)$ in $(R_s, 0)$, and to continue the orbit with smoothness, is perhaps the most natural choice — see Example 1 which presents the same local behavior.

[5]This does not mean that the limit of ρ as (R, H) tends to $(R_s, 0)$ exists, which is false.

Chapter 5

Conclusion

In the present work, motivated by some cosmological models of modified theories of gravity, we studied singular autonomous systems of ODEs which are now proposed in many physical, engineering (see [RB03], [CD89] and the references within), and demographic models (such as [BS19]). In Chapter 2, we addressed the planar case and specifically the form (2.2), which also incorporates the particular cosmological model (4.30). Inspired by Arnold's geometric theory of differential equations [Arn12], we studied the Carathéodory solutions of the singular system (2.2) by defining and analyzing a *regular system associated* with (2.2); remarkably, contrary to the geometric theory of differential equations, this approach can be generalized for d–dimensional, singular, autonomous systems of ODE of the form (2.33), as we discussed briefly in Section 2.3.

Observing that the singular points of (2.2) correspond to equilibria of the regular system (2.15), allows us to prove that in a generic case, the Carathéodory solution of (2.2) can cross the singular line if and only if evolving through a singular point — cfr. Proposition 1. Moreover, this construction permits us to describe some peculiar behaviours of solutions near singular points, such as when the latter ones are *funnel points* — we gave sufficient conditions for this to be the case in Theorem 1.

After discussing the historical and physical motivations supporting the introduction of modified theories of gravity, we addressed the case of metric $f(R)$ theories in Chapter 3, and, in particular, homogeneous and isotropic solutions of field equations as the Friedmann–Lemaître–Robertson–Walker (FLRW) universe. The latter, in the absence of matter and in the spatially–flat case, is described by a two–dimensional singular system of ODEs (4.30) where the dynamical variables are the Ricci scalar curvature R, and the Hubble parameter H. As the system (4.30) is in the form (2.19), we performed the general analysis that we developed in Chapter 2 to analyze it. In detail, under the assumptions that f is $C^3(I)$, and that $f_{RR} > 0$, i.e. that it is convex in the open interval $I \subseteq \mathbb{R}$ in order to avoid the Dolgov–Kawasaki instability, in Chapter 4 we proved that the system (4.30) has at most two proper singular points (in the sense of Definition 2.1.1). Moreover, if it also happens that $f(0) \neq 0$, as in the presence of a cosmological constant, we showed that the open semi–planes of the (R, H)–plane (where respectively $R > 0$ or $R < 0$) contain at most one singular point, and in this case they are *nondegenerate funnel points* of the first kind.

Moreover, we motivated how the dynamical importance of singular points is not only local, by showing that in some cases — such as in some *bounded curvature models* where I is an open bounded interval of reals — the solutions originating in the semi–plane where $H < 0$ are attracted by the singular point $(R_s, 0)$. Ultimately, this is a consequence of the fact that singular points correspond to equilibria of the regular associated system, which can therefore possess basins of attraction.

We concluded our dissertation considering the additional condition of continuing a Carathéo-

dory solution past the singular points keeping the incoming direction in (R, H)–phase space. As discussed in general in Chapter 2, this condition corresponds to the prescription of continuing the solution curve smoothly in the space of 1–jets with coordinates (x, y, p). Moreover, since under our assumptions the singular points are funnels of the first kind — i.e. the derivative of the vector field of the regular associated system has equal, real, eigenvalues of geometric multiplicity 2 there — this condition uniquely identifies a continuation of the incoming Carathéodory solution, and so with this additional prescription at the singularity one recovers the uniqueness of Initial Value Problems (IVPs); hence, it is possible to define a continuous dynamical system where (4.30) is well–defined.

Finally, on the one hand, this work intended to provide some geometric and qualitative results to describe singular points in autonomous systems of ODEs, treating the case of metric $f(R)$ FLRW cosmological models as an example, but, on the other hand, we wanted also to provide some insights on these models. In particular, we highlighted the presence of *bouncing cosmological solutions* and the importance of the *singular points* in the system (4.30), which are usually ignored, also by restricting to other regions of the phase space not containing them (see [dSF07]). Indeed, depending on the specific model and particular assumptions on f, as we showed in the Chapter 4 in the case of some *bounded curvature models*, the dynamical relevance of singular points is not only local, as they attract (bouncing) solutions originating in an open sub–domain of the phase space.

Appendix A

Equivariant and Reversible Dynamical Systems

We briefly recall here some classic definition and results about dynamical system which present some sort of symmetry in their trajectories. These systems have been largely studied by several authors thanks to their wide diffusion in various fields of mathematics. A quite complete survey that summarizes the state of the art and provides a rich bibliography including examples coming from both theoretical and applied mathematics is [LR98].

Throughout this appendix and, indeed, throughout the entirety of this work, we consistently make reference to the theory of dynamical systems, for a more comprehensive exploration of which we direct the reader to [Hal09], and [HK12] for the details. In particular, we will focus on autonomous differential equations of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t)), \tag{A.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is an open subset of $\mathbb{R}^d$, and $\boldsymbol{F} : \Omega \to \mathbb{R}^d$ is a continuous vector field. Recall that (A.1), under assumptions of existence and uniqueness of solution $\boldsymbol{x}(t)$ for the initial value problem (IVP):

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t)), \quad t \in I \subseteq \mathbb{R} \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \Omega, \end{cases} \tag{A.2}$$

determines a (local) dynamical system on the state space Ω by defining the one–parameter family of evolution operators $\{\varphi_t\}_{t\in\mathbb{R}}$ as

$$\begin{aligned} \varphi_t : \quad \Omega \quad &\longrightarrow \quad \Omega \\ \boldsymbol{x}(\tau) &\longmapsto \varphi_t(\boldsymbol{x}(\tau)) := \boldsymbol{x}(\tau + t). \end{aligned} \tag{A.3}$$

Indeed, under uniqueness assumptions, the family $\{\varphi_t\}_{t\in\mathbb{R}}$ is a group with respect to composition

$$\varphi_0 = id_\Omega, \quad \text{and} \quad \varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2} \quad \text{for all } t_1, t_2 \in \mathbb{R}.$$

A.1 Equivariant and reversible systems

Let us recall that

Definition A.1.1. A continuous and invertible map $\boldsymbol{S} : \Omega \to \Omega$ is a **symmetry** (or **equivariance**) of (A.1) when

$$\frac{d}{dt}(\boldsymbol{S}(\boldsymbol{x}(t)) = \boldsymbol{F}(\boldsymbol{S}(\boldsymbol{x}(t))), \tag{A.4}$$

and in this case the original system is called **S–equivariant**. In terms of the evolution operators, condition above is equivalent to

$$\boldsymbol{S} \circ \varphi_t = \varphi_t \circ \boldsymbol{S} \quad \text{for all} \quad t \in \mathbb{R}. \tag{A.5}$$

The latter is perhaps the definition that better fits in the abstract setting of dynamical systems, since it only involves the family of evolution operators and so can be easily stated for both discrete and continuous–time systems.

Definition A.1.2. A continuous and invertible map $\boldsymbol{G} : \Omega \to \Omega$ is a **reversing symmetry** of (A.1) when

$$\frac{d}{dt}(\boldsymbol{G}(\boldsymbol{x}(t)) = -\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{x}(t))). \tag{A.6}$$

Again, in terms of the evolution operators, condition above is equivalent to

$$\boldsymbol{G} \circ \varphi_t = \varphi_{-t} \circ \boldsymbol{G} = \varphi_t^{-1} \circ \boldsymbol{G} \quad \text{for all} \quad t \in \mathbb{R}. \tag{A.7}$$

If in addition $\boldsymbol{G}$ is an involution, i.e. $\boldsymbol{G} \circ \boldsymbol{G} = id_\Omega$, it is called an **involutory reversing symmetry**.

From a dynamical prospective, equivariances map trajectories into other trajectories preserving direction in which they are traversed in time, while the reversing symmetries do the same inverting the "time–arrow".

Remark 8. Let us denote the components of a two–dimensional vector field $\boldsymbol{F} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\boldsymbol{F}\left(x, y\right) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}. \tag{A.8}$$

If we consider the involution determined by the reflection about the x–axis, i.e. $\boldsymbol{G}(x, y) = (x, -y)$, then $\boldsymbol{F}$ is reversible (relatively to $\boldsymbol{G}$) if and only if its components are respectively odd and even with respect to y, i.e.

$$F_1(x, -y) = -F_1(x, y) \quad \text{and} \quad F_2(x, -y) = F_2(x, y).$$

Conversely, if the components are respectively even and odd with respect to y, the reflection about the x–axis is a symmetry of (A.1).

Example 10. In our work we study the system

$$\begin{cases} \dot{x} &= f_x(x)(x - 6y^2) - f(x) \\ \dot{y} &= y \cdot f_{xx}(x)(x - 12y^2), \end{cases} \tag{A.9}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a fixed function in $C^3(I)$. By applying the considerations above, we conclude that the system (A.9) is symmetric with respect to the reflection over the x–axis.

Example 11. Another useful consequence of the above remark[1] is that Hamiltonian systems possess the anti–symplectic, involutory reversing symmetry $\boldsymbol{G}(\boldsymbol{q}, \boldsymbol{p}) := (\boldsymbol{q}, -\boldsymbol{p})$ if their Hamiltonian satisfies

$$H(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{q}, -\boldsymbol{p}). \tag{A.10}$$

Indeed from (A.10) we find

$$\frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q}, -\boldsymbol{p}) = -\frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q}, \boldsymbol{p}), \qquad \frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{q}, -\boldsymbol{p}) = \frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}),$$

[1] Easily generalizable in dimension $2d$.

and by letting

$$\begin{cases} \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q},\boldsymbol{p}) =: F_1(\boldsymbol{q},\boldsymbol{p}) \\ \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p}) =: F_2(\boldsymbol{q},\boldsymbol{p}) \end{cases}$$

we have that $F_1(\boldsymbol{q},-\boldsymbol{p}) = -F_1(\boldsymbol{q},\boldsymbol{p})$ and $F_2(\boldsymbol{q},-\boldsymbol{p}) = F_2(\boldsymbol{q},\boldsymbol{p})$.

It is easy to show (and not surprising) that the symmetries of a dynamical system are closed under composition and moreover form a group $\mathtt{G}$. In particular, the composition of equivariances or an even number of reversing symmetries is an equivariance, while the composition of an odd number of reversing symmetries is a reversing symmetry.

In other words, the equivariances form a normal subgroup $\mathtt{S} \leq \mathtt{G}$ which has either index one or two, and therefore if there are no non–trivial equivariances $\mathtt{G}$ is either trivial or $\mathtt{G} \cong \mathbb{Z}/2\mathbb{Z}$. In the latter case the system has exactly one reversing symmetry and therefore is sometimes called **purely reversible**.

Proposition 2. *The system possesses at least one involution if and only if we can write $\mathtt{G}$ as the semi–direct product $\mathtt{G} \cong \mathtt{H} \rtimes \mathbb{Z}/2\mathbb{Z}$.*

A.2 Symmetric periodic orbits

Ever since the work of Poincaré, the presence of periodic orbits is one of the most studied features of a dynamical system. In the particular case of reversible dynamical systems, much progresses were made by Birkhoff who, in 1915 [Bir15], used reversibility to find and analyze periodic orbits in the restricted three–body problem. In 1958 DeVogelaere reformulated this method and wrote an algorithm to let a computer find symmetric periodic orbits of reversible systems, in the sense of this definition:

Definition A.2.1. Let $o(\boldsymbol{x}) := \{\varphi_t(\boldsymbol{x}) \mid t \in \mathbb{R}\}$ be an orbit of a dynamical system. We say that $o(\boldsymbol{x})$ is **G–symmetric** or **symmetric with respect to G** when the orbit is set–wise invariant under $\boldsymbol{G}$, in other terms when $\boldsymbol{G}(o(\boldsymbol{x})) = o(\boldsymbol{x})$.

The main result regarding symmetric orbits, which remarkably does not require G to be an involution, is the following one (see [Lam92] for the details and the proofs of the following results).

Theorem 3. *Let $o(\boldsymbol{x})$ be an orbit of the flow of an autonomous vector field with time–reversal symmetry $\boldsymbol{G}$. Then*

(i) An orbit $o(\boldsymbol{x})$ is symmetric with respect to $\boldsymbol{G}$ if and only if $o(\boldsymbol{x})$ intersects $\mathrm{Fix}(\boldsymbol{G})$, in which case the orbit intersects $\mathrm{Fix}(\boldsymbol{G})$ in no more than two points and is fully contained in $\mathrm{Fix}(G^2)$.

(ii) An orbit $o(\boldsymbol{x})$ intersects $\mathrm{Fix}(\boldsymbol{G})$ in exactly two points if and only if it is periodic (and it is not a fixed point) and symmetric with respect to $\boldsymbol{G}$.

Previous result provides a method that can be implemented to find $\boldsymbol{G}$–symmetric periodic orbits [Greene 1981]. Moreover, the latter suggests that these orbits typically arise in families, which under hypothesis of smoothness usually happen to be manifolds.

Theorem 4 (Families of symmetric periodic orbits)**.** *In a continuous m–parameter family of autonomous flows with reversing symmetry $\boldsymbol{G}$*

(i) $\boldsymbol{G}$–symmetric periodic orbits of a given period typically arise as $(2\dim \mathrm{Fix}(\boldsymbol{G}) - \dim \mathrm{Fix}(\boldsymbol{G}^2) + m)$–parameter families.

(ii) $\boldsymbol{G}$*–symmetric periodic orbits usually arise as* $(2\dim \mathrm{Fix}(\boldsymbol{G}) - \dim \mathrm{Fix}(\boldsymbol{G}^2) + m + 1)$*–parameter families with smoothly varying period, in which the families with constant period above are embedded.*

Appendix B

Geometric theory of differential equations

In this chapter, following [Arn12], we briefly present the basic notions of the geometric theory of implicit that are not solved for the derivative.

B.1 Implicit differential equations

Consider a generic differential equation implicit in the derivative as

$$G(x, y, p) = 0, \tag{B.1}$$

where $p = dy/dx$ and G is a smooth function in some domain of $\mathbb{R}^3$.

The three–dimensional space with coordinates (x, y, p) is called the *space of 1–jets of the function* $y(x)$, as two different functions $y_1(x)$ and $y_2(x)$ are said to have the same k–jet at a point x_0 if

$$|y_1(x) - y_2(x)| = \mathcal{O}(|x - x_0|^k). \tag{B.2}$$

Notice that, by the implicit function theorem, provided the total differential of G is not zero at some point, the surface $\mathscr{M}$ defined by (B.1) in the three–dimensional space of 1–jets with coordinates (x, y, p) is smooth, and later on we will assume that this is always the case.

In such a surface, it is possible to construct a direction field defined as follows. First of all, we denote the components with respect to the coordinates (x, y, p) of a vector ξ as $(dx(\xi), dy(\xi), dp(\xi))$. Then, at each point (x, y, p), we consider the plane composed by the vectors such that

$$dy(\xi) = p\, dx(\xi), \tag{B.3}$$

and call it *contact plane*. Notice that such a vector ξ satisfies (B.3), if its projection on the (x, y)–plane forms an angle with respect to the x–axis whose tangent is p (see Figure B.1).

In this way, we can define a *contact field of planes*, i.e. a field of 1–forms, in the whole space of 1–jets. Moreover, at each point (x_0, y_0, p_0) of the smooth surface $\mathscr{M}$, the contact plane and the plane tangent to $\mathscr{M}$ at (x_0, y_0, p_0) intersect into a straight line (if they do not coincide).

Definition B.1.1. The *integral curves* of (B.1) are the integral curves of the direction field defined by intersecting the contact and tangent plane at each point of $\mathscr{M}$.

The relation between the integral curves of (B.1) and the solutions of the latter in the (x, y)–plane is explored in Theorem 5.

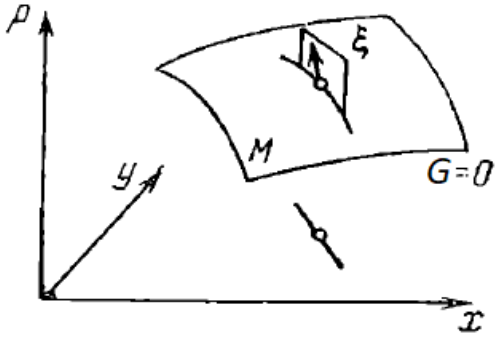

Figure B.1: Contact plane at point (x, y, p).

Definition B.1.2. A point of the surface $\mathscr{M}$ is said to be *regular* if it is not a critical point of the vertical projection mapping

$$\pi : \mathscr{M} \to \mathbb{R}^2, \quad \pi(x, y, p) = (x, y). \tag{B.4}$$

Equivalently, a point on $\mathscr{M}$ is *regular* if the tangent plane to $\mathscr{M}$ at this point is not vertical. Near regular points, there is a correspondence between integral curves as defined geometrically above, and the solutions of the differential equation (B.1). In fact, by the implicit function theorem, there is a neighborhood of such regular point (x_0, y_0, p_0), where the surface $\mathscr{M}$ can be parametrized as the graph of a function $p = v(x, y)$.

Theorem 5. *The projection onto the (x, y)–plane maps the integral curves of* (B.1) *on $\mathscr{M}$ (as in Definition B.1.1) in the neighborhood of a regular point, into the integral curves of the differential equation*

$$\frac{dy}{dx} = v(x, y), \tag{B.5}$$

in the neighborhood of the projection of this point (see Figure B.2).

Proof. By definition, the projection of the contact plane at a point (x_0, y_0, p_0) onto the (x, y)–plane is a straight line with direction equal to $v(x_0, y_0)$. The thesis follows by the fact that π is a local diffeomorphism at (x_0, y_0, p_0). □

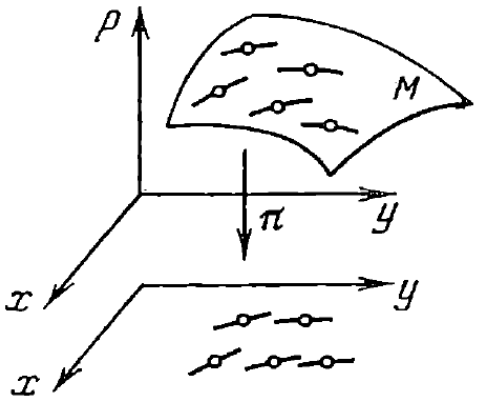

Figure B.2: Projection of integral curves of (B.1) on $\mathscr{M}$ onto the (x, y)–plane.

B.2 Singular points

At this point, we can finally give the following important definition:

Definition B.2.1. A point of the surface $\mathscr{M}$ is said to be *singular* if it is a critical point of the vertical projection mapping, i.e. if π is not a local diffeomorphism in the neighborhood of this point.

Notice that by the implicit function theorem, the singular points are the ones where $G = 0$ and $G_p = 0$, so in general, they form a curve in the three–dimensional space of 1–jets which is called *criminant curve*:

$$\texttt{CC} = \{(x, y, p) \mid G(x, y, p) = 0, \quad G_p(x, y, p) = 0\}. \tag{B.6}$$

The projection of the latter onto the (x, y)–plane is called *discriminant curve*:

$$\texttt{DC} = \pi[\texttt{CC}]. \tag{B.7}$$

Remark 9. When such a projection is performed, even if the criminant is everywhere smooth, geometric singularities may arise, as the map is not injective. Generically, these correspond to points of self–intersection of the discriminant curve, having two or more branches intersecting with some angle on the (x, y)–plane. For more details about these features, we refer to [Arn12], and [Dav85].

Moreover, we point out that the criminant is a smooth curve in the space of 1–jets in the neighborhood of its points where the rank of the derivative of the mapping

$$\Phi : (x, y, p) \mapsto (G, G_p) \tag{B.8}$$

is maximal, i.e. equal to 2. This remark suggests the following:

Definition B.2.2. A singular point of (B.1) is said to be *regular* if at this point the derivative of the map (B.8) has rank 2, and if the criminant is not tangent to the contact plane at that point.

Note that in the generic case, almost all the singular points are regular, and the non–regular ones are discrete in CC. If this is not the case, it is possible to consider a small perturbation of the vector field and reduce it to the above–mentioned situation.

Example 12. If we consider

$$G(x, y, p) = p^2 - x = 0, \tag{B.9}$$

the criminant curve is the y–axis, which coincides with its projection, i.e. the discriminant curve. The condition of smoothness of the criminant is verified, as at singular points we have

$$D\Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{B.10}$$

which has rank 2. In addition, the tangent vector to the criminant, i.e. $\xi = (0, 1, 0)$, does not belong to the contact plane, which at singular points is defined by $dy = 0\, dx$. We deduce that all the singular points of the equation $p^2 = x$ are regular (see Figure B.3).

It is possible to show that the integral curves in a neighborhood of a regular singular point of (B.1) are diffeomorphic to those of the equation $p^2 = x$, which are semi–cubic parabolas of equation $y = x^{3/2} + C$, with $C \in \mathbb{R}$.

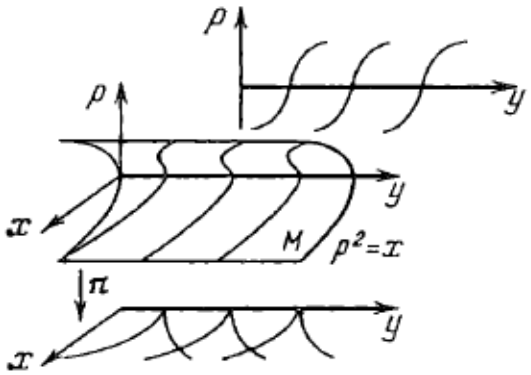

Figure B.3: Integral curves of the equation (B.9).

Theorem 6 (Normal form)**.** *Let (x_0, y_0, p_0) be a regular singular point of the equation* (B.1)*. Then, there exists a diffeomorphism of the neighborhood of the point (x_0, y_0) in the (x, y)–plane onto the neighborhood of the point $(0, 0)$ in the (X, Y)–plane which reduces the equation $G = 0$ to the form $P^2 = X$, with $P = dY/dX$.*

Proof. See Chapter 1.4 of [Arn12]. □

As we shall see in the next section, there are cases where the criminant and the discriminant curve are smooth, but there are singular points where the contact plane to $\mathscr{M}$ contains the tangent vector to CC. It is possible to show, see [Dav85], that in a neighborhood of such a point, one can reduce $G = 0$ to the normal form

$$y = (p + kx)^2, \tag{B.11}$$

by applying a diffeomorphsim of the (x, y)–plane.

Example 13. It is easy to check that the criminant curve of (B.11) is

$$\mathtt{CC} = \{(x, y, p) \mid p = -kx, \quad y = 0\}, \tag{B.12}$$

therefore its tangent vector is $\xi = (1, 0, -k)$, and the discriminant curve is the x–axis. The condition of smoothness of criminant is verified, as at singular points

$$D\Phi = \begin{pmatrix} 2k(p + kx) & -1 & 2(p + kx) \\ 2k & 0 & 2 \end{pmatrix} \tag{B.13}$$

which has rank 2. However, at the origin, the tangent vector ξ is contained in the contact plane $dy = 0\,dx$, and so it is the only singular point which is not regular.

At a generic point where the criminant is tangent to the contact plane, the equation (B.1) has a singularity of the same type of the equilibrium of a planar system — i.e. a saddle, a node, or a focus. For this reason, the corresponding singular points of (B.1), obtained through projection onto the (x, y)–plane, are called *folded saddle, nodes, or foci* (see Figure B.4).

B.3 Rational vector fields

In the spirit of Chapter 2, we are interested in studying the singular points of the autonomous, planar system:

$$\begin{cases} \alpha(x, y)\ \dot{x} = \beta(x, y) \\ \gamma(x, y)\ \dot{y} = \delta(x, y). \end{cases} \tag{B.14}$$

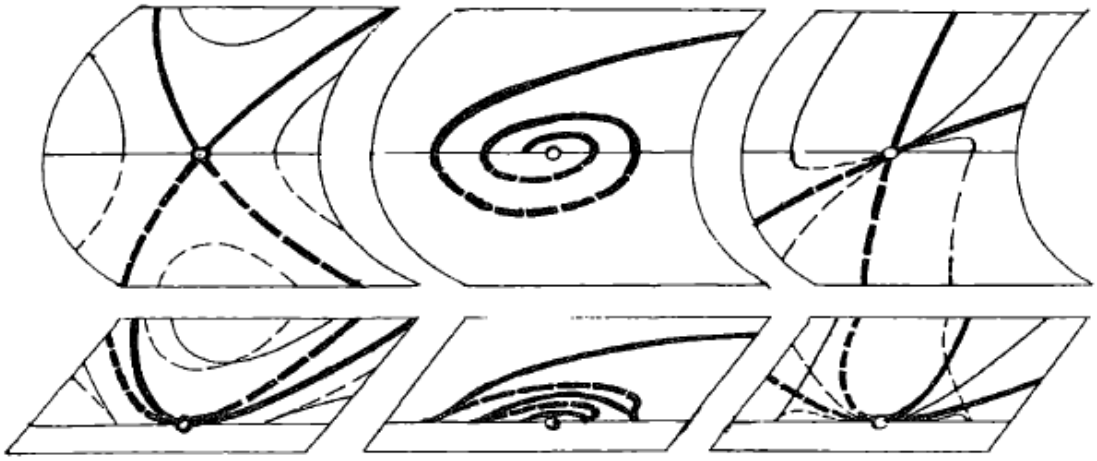

Figure B.4: Respectively a folded saddle, focus and node.

We can view the solutions of (B.14) as a particular[1] parametrization of the integral curves of the implicit differential equation

$$G(x, y, p) = \beta(x, y)\gamma(x, y)p - \alpha(x, y)\delta(x, y) = 0, \tag{B.15}$$

as

$$p = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\alpha(x, y)\delta(x, y)}{\beta(x, y)\gamma(x, y)}. \tag{B.16}$$

The criminant curve of (B.15) is

$$\mathtt{CC} = \{(x, y, p) \mid \beta(x, y)\gamma(x, y) = 0, \quad \alpha(x, y)\delta(x, y) = 0\}, \tag{B.17}$$

which, in the generic case, is composed of straight lines parallel to the p–axis. It follows that the discriminant curve is formed by isolated points. In addition, as the tangent vector to $\mathtt{CC}$ is $\xi = (0, 0, 1)$ which belongs to the contact plane, the singular points are never regular even if

$$D\Phi = \begin{pmatrix} \frac{\partial(\beta\cdot\gamma)}{\partial x}p - \frac{\partial(\alpha\cdot\delta)}{\partial x} & \frac{\partial(\beta\cdot\gamma)}{\partial y}p - \frac{\partial(\alpha\cdot\delta)}{\partial y} & \beta\cdot\gamma \\ \\ \frac{\partial(\beta\cdot\gamma)}{\partial x} & \frac{\partial(\beta\cdot\gamma)}{\partial y} & 0 \end{pmatrix}, \tag{B.18}$$

has maximal rank.

Finally, we point out that the fact that these singular points correspond to folded saddle, nodes, or foci — as said in the above section — is coherent with the fact that they correspond to equilibria of the system

$$\begin{cases} \dot{x} = \beta(x, y)\gamma(x, y) \\ \dot{y} = \delta(x, y)\alpha(x, y). \end{cases} \tag{B.19}$$

[1]Notice, for instance, that the solutions of (B.14) and those of its regular associated system (as defined in Chapter 2) are two different parametrizations of the same integral curves.

Appendix C

Addendum

In this part we present the computations for the Taylor expansion of the right–hand side in equation (4.1).

C.1 Taylor expansion of $\rho(R, H)$

We present here the Taylor expansion of the vector field first component $\rho(R, H)$ in the neighborhood of a non–generic point $\bar{P} = (\bar{R}, 0)$. Let $\Delta R := R - \bar{R}$ and

$$f(R) = \bar{f} + \bar{f}_R \Delta R + \frac{1}{2}\bar{f}_{RR}(\Delta R)^2 + \mathcal{O}((\Delta R)^3), \tag{C.1}$$

$$f_R(R) = \bar{f}_R + \bar{f}_{RR}\Delta R + \frac{1}{2}\bar{f}_{RRR}(\Delta R)^2 + \mathcal{O}((\Delta R)^3), \tag{C.2}$$

$$f_{RR}(R) = \bar{f}_{RR} + \bar{f}_{RRR}\Delta R + \frac{1}{2}\bar{f}_{RRRR}(\Delta R)^2 + \mathcal{O}((\Delta R)^3), \tag{C.3}$$

where we used the notation $\bar{g} := g(\bar{R})$, to write functions of R evaluated in $\bar{R}$. Using the expansions above allows us to write

$$f_R(R)R - f(R) = (\bar{R}\bar{f}_R - \bar{f}) + \bar{R}\bar{f}_{RR}\Delta R + \frac{1}{2}(\bar{R}\bar{f}_{RRR} + \bar{f}_{RR})(\Delta R)^2 + \mathcal{O}(\Delta R)^3), \tag{C.4}$$

$$f_{RR}(R)^{-1} = \bar{f}_{RR}^{-1}\left(1 - \frac{\bar{f}_{RRR}}{\bar{f}_{RR}}\Delta R - \frac{1}{2}\frac{\bar{f}_{RRRR}}{\bar{f}_{RR}}(\Delta R)^2\right) + \mathcal{O}((\Delta R)^3), \tag{C.5}$$

and

$$\begin{aligned}\frac{f_R(R)R - f(R)}{f_{RR}(R)} =& \frac{\bar{f}_R\bar{R} - \bar{f}}{\bar{f}_{RR}} + \left(\bar{R} - \left(\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}\right)\left(\frac{\bar{f}_R\bar{R} - \bar{f}}{\bar{f}_{RR}}\right)\right)\Delta R + \\ &+ \frac{1}{2}\left(1 - \bar{R}\frac{\bar{f}_{RRR}}{\bar{f}_{RR}} - \left(\frac{\bar{f}_{RRRR}}{\bar{f}_{RR}}\right)\left(\frac{\bar{f}_R\bar{R} - \bar{f}}{\bar{f}_{RR}}\right)\right)(\Delta R)^2 + \mathcal{O}((\Delta R)^3),\end{aligned} \tag{C.6}$$

$$\frac{f_R(R)}{f_{RR}(R)} = \frac{\bar{f}_R}{\bar{f}_{RR}} + \left(1 - \frac{\bar{f}_{RRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)\Delta R - \frac{1}{2}\left(\frac{\bar{f}_{RRR}}{\bar{f}_{RR}} + \frac{\bar{f}_{RRRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)(\Delta R)^2 + \mathcal{O}((\Delta R)^3). \tag{C.7}$$

Substituting the above equations into (4.1) gives

$$\rho(R, H) = \frac{\bar{f}_R\bar{R} - \bar{f} - 6\bar{f}_R H^2}{6H\bar{f}_{RR}} + \left[\frac{1}{6H}\left(\bar{R} - \frac{\bar{f}_{RRR}}{\bar{f}_{RR}}\right)\left(\frac{\bar{f}_R\bar{R} - \bar{f}}{\bar{f}_{RR}}\right) - \left(1 - \frac{\bar{f}_{RRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)H\right]\Delta R +$$

$$+\left[\frac{1}{12H}\left(1-\bar{R}\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}-\left(\frac{\bar{f}_{RRRR}}{\bar{f}_{RR}}\right)\left(\frac{\bar{f}_R\bar{R}-\bar{f}}{\bar{f}_{RR}}\right)\right)+\right.$$
$$\left.-\frac{1}{2}\left(\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}+\frac{\bar{f}_{RRRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)H\right](\Delta R)^2+\mathcal{O}((\Delta R)^3). \tag{C.8}$$

Notice that the condition (4.25) must hold in order the above quantity not to diverge in the limit $H\to 0$, when $\bar{R}$ is in the open interval (R_l,R_r), where we are assuming that f_{RR} is finite. Under this condition, (C.8) reads

$$\rho(R,H)=-\frac{\bar{f}_R}{\bar{f}_{RR}}H+\left[\frac{\bar{R}}{6H}-\left(1-\frac{\bar{f}_{RRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)H\right]\Delta R+$$
$$+\left[\frac{1}{12H}\left(1-\bar{R}\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}\right)-\frac{1}{2}\left(\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}+\frac{\bar{f}_{RRRR}\bar{f}_R}{\bar{f}_{RR}^2}\right)H\right](\Delta R)^2+\mathcal{O}((\Delta R)^3). \tag{C.9}$$

We deduce that, as remarked before, the limit of $\rho(R,H)$ for $R\to\bar{R}$ and $H\to 0$ does not exist. In fact, approaching $\bar{P}$ along different directions by choosing $H=\alpha\Delta R$ gives different limits

$$\rho(\bar{R}+\Delta R,\alpha\Delta R)=\frac{R}{6\alpha}+\frac{1}{12\alpha}\left[1-\bar{R}\frac{\bar{f}_{RRR}}{\bar{f}_{RR}}-12\alpha^2\frac{\bar{f}_R}{\bar{f}_{RR}}\right]\Delta R+\mathcal{O}((\Delta R)^2). \tag{C.10}$$

Bibliography

[A+06] P. Astier et al. The supernova legacy survey: measurement of Ω_M, Ω_Λ, and w from the first year data set. *Astronomy & Astrophysics*, 447(1):31–48, 2006.

[AGPT07] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa. Conditions for the cosmological viability of $f(R)$ dark energy models. *Physical Review D*, 75(8):083504, 2007.

[Arn12] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*, volume 250. Springer Science & Business Media, Berlin, DE, 2012.

[BD84] N. D. Birrell and P. C. W. Davies. *Quantum fields in curved space*. Cambridge university press, Cambridge, UK, 1984.

[Bek04] J. D. Bekenstein. Relativistic gravitation theory for the modified Newtonian dynamics paradigm. *Physical Review D*, 70(8):083509, 2004.

[Ber68] P. G. Bergmann. Comments on the scalar-tensor theory. *International Journal of Theoretical Physics*, 1:25–36, 1968.

[Bir09] G. D. Birkhoff. Singular points of ordinary linear differential equations. *Transactions of the American Mathematical Society*, 10(4):436–470, 1909.

[Bir15] G. D. Birkhoff. The restricted problem of three bodies. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 39(1):265–334, 1915.

[BOS92] I. L. Buchbinder, S. Odintsov, and L. Shapiro. *Effective Action in Quantum Gravity*. CRC Press, Boca Raton, FL, 1992.

[BS19] B. Babaei and M. Shafiee. Analysis and behavior control of a modified singular prey–predator model. *European Journal of Control*, 49:107–115, 2019.

[Car01] S. M. Carroll. The cosmological constant. *Living Reviews in Relativity*, 4(1):1–56, 2001.

[CCT05] S. Capozziello, V. F. Cardone, and A. Troisi. Reconciling dark energy models with $f(R)$ theories. *Physical Review D*, 71(4):043503, 2005.

[CD89] I. O. Chua and A. Deng. Impasse points. Part I: numerical aspects. *International Journal of Circuit Theory and Applications*, 17(2):213–235, 1989.

[CL03] P. Coles and F. Lucchin. *Cosmology: The Origin and Evolution of Cosmic Structure*. John Wiley & Sons, Hoboken, NJ, 2003.

[Cor08] J. Cortes. Discontinuous Dynamical Systems. *IEEE Control systems magazine*, 28(3):36–73, 2008.

[Dav85] A.A. Davydov. The normal form of the differential equation implicit in the derivative. *Functional Analysis and its Applications*, 19(2):81–89, 1985.

[DGP00] G. Dvali, G. Gabadadze, and M. Porrati. 4d gravity on a brane in 5d Minkowski space. *Physics Letters B*, 485(1-3):208–214, 2000.

[DK03] A. D. Dolgov and M. Kawasaki. Can modified gravity explain accelerated cosmic expansion? *Physics Letters B*, 573:1–4, 2003.

[dSF07] J. de Souza and V. Faraoni. The phase-space view of $f(R)$ gravity. *Classical and Quantum Gravity*, 24(14):3637, 2007.

[EGS68] J. Ehlers, P. Geren, and R. Sachs. Isotropic solutions of the Einstein-Liouville equations. *Journal of Mathematical Physics*, 9(9):1344–1349, 1968.

[EHC08] J. D. Evans, L. Hall, and P. Caillol. Standard cosmological evolution in a wide range of $f(R)$ models. *Physical Review D*, 77(8):083514, 2008.

[Far06] V. Faraoni. Matter instability in modified gravity. *Physical Review D*, 74(10):104017, 2006.

[Fil13] A. F. Filippov. *Differential Equations with Discontinuous Right–hand Sides: Control Systems*, volume 18. Springer Science & Business Media, Berlin, DE, 2013.

[FZ21] V. P. Frolov and A. Zelnikov. Bouncing cosmology in the limiting curvature theory of gravity. *Physical Review D*, 104(10):104060, 2021.

[Gal81] G. Gale. The anthropic principle. *Scientific American*, 245(6):154–171, 1981.

[GCT10] A. Guarnizo, L. Castaneda, and J. M. Tejeiro. Boundary term in metric $f(R)$ gravity: field equations in the metric formalism. *General Relativity and Gravitation*, 42:2713–2728, 2010.

[Hal09] J. K. Hale. *Ordinary differential equations*. Courier Corporation, North Chelmsford, Massachusetts, 2009.

[HK12] J. K. Hale and H. Koçak. *Dynamics and bifurcations*, volume 3. Springer Science & Business Media, 2012.

[Lam92] J. Lamb. Reversing symmetries in dynamical systems. *Journal of Physics A: Mathematical and General*, 25(4):925, 1992.

[LR98] J. Lamb and J. Roberts. Time-reversal symmetry in dynamical systems: a survey. *Physica D: Nonlinear Phenomena*, 112(1-2):1–39, 1998.

[MS20] H. Motohashi and T. Suyama. Quantum Ostrogradsky theorem. *Journal of High Energy Physics*, 2020(9):1–10, 2020.

[MTW73] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. Macmillan, London, 1973.

[NGD14] A. Nzioki, R. Goswami, and P. Dunsby. Jebsen–Birkhoff theorem and its stability in $f(R)$ gravity. *Physical Review D*, 89(6):064050, 2014.

[NO03] S. Nojiri and S. D. Odintsov. Where new gravitational physics comes from: M-theory? *Physics Letters B*, 576(1-2):5–11, 2003.

[NO06] S. Nojiri and S. D. Odintsov. Modified $f(R)$ gravity consistent with realistic cosmology: From a matter dominated epoch to a dark energy universe. *Physical Review D*, 74(8):086005, 2006.

[PSTK94] P. J. Peebles, D. N. Schramm, E. L. Turner, and R. G. Kron. The evolution of the universe. *Scientific American*, 271(4):52–57, 1994.

[R+04] A. G. Riess et al. Type Ia supernova discoveries at $z > 1$ from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution. *The Astrophysical Journal*, 607(2):665, 2004.

[RB03] G. Reißig and H. Boche. On singularities of autonomous implicit ordinary differential equations. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 50(7):922–931, 2003.

[RvETT96] S. Rippl, H. van Elst, R. Tavakol, and D. Taylor. Kinematics and dynamics of $f(R)$ theories of gravity. *General Relativity and Gravitation*, 28:193–205, 1996.

[S+17] P.K. Sahoo et al. Magnetized strange quark model with big rip singularity in $f(R,T)$ gravity. *Modern Physics Letters A*, 32(21):1750105, 2017.

[SA] Y. Sakakihara S. Ansoldi, H. Ishihara. Private communication.

[Sch07] H. J. Schmidt. Fourth order gravity: Equations, history, and applications to cosmology. *International Journal of Geometric Methods in Modern Physics*, 4(02):209–248, 2007.

[SF10] T. P. Sotiriou and V. Faraoni. $f(R)$ theories of gravity. *Reviews of Modern Physics*, 82(1):451, 2010.

[Sol77] L. Solomon. *Differential Equations: Geometric Theory.* Dover, New York, NY, 1977.

[Sta80] A. A. Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1):99–102, 1980.

[Ste77] K. S. Stelle. Renormalization of higher-derivative quantum gravity. *Physical Review D*, 16(4):953, 1977.

[SZ18] H. Shabani and A.H. Ziaie. Bouncing cosmological solutions from $f(R,T)$ gravity. *The European Physical Journal C*, 78:1–24, 2018.

[UD62] R. Utiyama and B. S. DeWitt. Renormalization of a classical gravitational field interacting with quantized matter fields. *Journal of Mathematical Physics*, 3(4):608–618, 1962.

[Vil92] G. A. Vilkovisky. Effective action in quantum gravity. *Classical and Quantum Gravity*, 9(4):895, 1992.

[Wal10] R. M. Wald. *General Relativity.* University of Chicago press, Chicago, IL, 2010.

[Wil18] C. M. Will. *Theory and Experiment in Gravitational Physics.* Cambridge university press, Cambridge, UK, 2018.

[Woo07] R. Woodard. Avoiding dark energy with $1/R$ modifications of gravity. In *The Invisible Universe: Dark Matter and Dark Energy*, pages 403–433. Springer, Berlin, DE, 2007.