Direct Method and Convexity in the Calculus of Variations: De Giorgi–Ioffe Theorem

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Introduction

We are interested in determining the existence of *minima* of functionals of the form

$$F(u) = \int_\Omega f(x, u(x), Du(x))\, dx, \qquad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m. \tag{1}$$

A *direct way* to approach such a problem is provided by Tonelli's theorem[1] which requires that $F : (X, \tau) \to \overline{\mathbb{R}}$ is **coercive** and **lower semicontinuous** (l.s.c.).

- The *coerciveness* is more of a "quantitative condition" and usually reduces into finding some estimates from below on F.
- The *lower semicontinuity* involves "qualitative" properties of the integrand $f(x, z, \xi)$ and in some situations may not hold.

[1] Also known as *"Direct Method in the Calculus of Variations"*.

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Direct Method in the Calculus of Variations

Some recalls

Definition

Let X be a topological space and $x \in X$. A function $f : X \to \overline{\mathbb{R}}$ is

- **lower semicontinuous** (l.s.c) if $f^{-1}((t, +\infty])$ is open in X for all $t \in \mathbb{R}$;
- **sequentially lower semicontinuous in the point** $x \in X$ if

$$f(x) \leq \liminf_{n \to \infty} f(x_n), \quad \text{for all } x_n \to x;$$

- **coercive** (resp. **sequentially coercive**) if for all $t \in \mathbb{R}$ exists a closed and compact (resp. sequentially compact) subset $K_t \subset X$ such that

$$\{x \in X : f(x) \leq t\} \subseteq K_t.$$

Direct Method in the Calculus of Variations

Theorem (Tonelli)

Let $F : (X, \tau) \to \overline{\mathbb{R}}$ *a function which is*

i) *lower semicontinuous (resp. sequentially lower semicontinuous);*

ii) *coercive (resp. sequentially coercive).*

Then F *admits minimum on* X*.*

Direct Method in the Calculus of Variations

Theorem (Tonelli)

Let $F : (X, \tau) \to \overline{\mathbb{R}}$ a function which is

i) *lower semicontinuous (resp. sequentially lower semicontinuous);*

ii) *coercive (resp. sequentially coercive).*

Then F admits minimum on X.

- It is therefore necessary to find a domain and a topology which is fine enough so to have l.s.c., but not too much so to have enough compact sets for the coerciveness.

Sufficient conditions for lower semicontinuity

Convexity of the integrand with respect to ξ

- As already remarked, it is not an easy task to prove the l.s.c. of the functional F; we are therefore interested in conditions that ensures such a property.
- In particular we will cover the case of integrand $f(x, z, \xi)$ which is "convex" with respect to ξ. This condition is relatable to several physical examples:

Example

Consider $F : W_0^{1,2}(\Omega) \to \overline{\mathbb{R}}$ defined as

$$F(u) = \int_\Omega \langle A(x)\nabla u, \nabla u\rangle \, dx - \int_\Omega g(x) u \, dx$$

and representing the total energy of a thermal conductor covering the region $\Omega \subset \mathbb{R}^n$, under the effect of several heat sources.

In order to formalize what said before and state the main results, we give:

Definition

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and μ be a positive, finite and complete measure on $\mathscr{F}$ a σ–algebra of Ω. A function $f : \Omega \times \mathbb{R}^m \to (-\infty, +\infty]$ is said to be

- an **integrand** if it is $\mathscr{F} \otimes \mathscr{B}_m$–measurable;
- a **normal integrand** if f is an integrand and $f(x, \cdot)$ is l.s.c. on $\mathbb{R}^m$ for μ–a.e. $x \in \Omega$;
- a **convex integrand** if f is an integrand and $f(x, \cdot)$ is convex and l.s.c. on $\mathbb{R}^m$ for μ–a.e. $x \in \Omega$.
- a **Carathéodory integrand** if it is an integrand and $f(x, \cdot)$ is finite and continuous on $\mathbb{R}^m$ for μ–a.e. $x \in \Omega$.

A function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to (-\infty, +\infty]$ is said to be

- a **normal–convex integrand** if it is $\mathscr{F} \otimes \mathscr{B}_m \otimes \mathscr{B}_k$–measurable and there exists $N \subset \Omega$ with $\mu(N) = 0$ such that
  i) $f(x, \cdot, \cdot)$ is l.s.c. on $\mathbb{R}^m \times \mathbb{R}^k$ for every $x \in \Omega \backslash N$.
  ii) $f(x, z, \cdot)$ is convex on $\mathbb{R}^k$ for every $x \in \Omega \backslash N$ and $z \in \mathbb{R}^m$.

Theorem (De Giorgi [1] – Ioffe [2])

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to [0, +\infty]$ a normal–convex integrand. Then the functional $F : L^1_\mu(\Omega; \mathbb{R}^m) \times L^1_\mu(\Omega; \mathbb{R}^k) \to [0, +\infty]$ defined as

$$F(u, v) = \int_\Omega f(x, u(x), v(x))\, d\mu(x),$$

is well–posed and l.s.c. with respect to the strong topology of $L^1_\mu(\Omega; \mathbb{R}^m)$ in the variable u, and the weak topology of $L^1_\mu(\Omega; \mathbb{R}^k)$ in the variable v.

Corollary

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to [0, +\infty]$ be a normal–convex integrand. Then the functional

$$F(u) = \int_\Omega f(x, u(x), Du(x))dx$$

is sequentially l.s.c. with respect to the weak topology of $W^{1,1}(\Omega; \mathbb{R}^m)$.

Outline of the proof

- From now on we will follow De Giorgi, Buttazzo and Dal Maso who later presented an alternative proof of the result [3].
- Their idea was to express the integrand f as the *supremum* of a countable family of functions which are easier to deal with.
- Finally, with some caution and machinery it will be possible to "drag the *supremum* out of the integral". Here the hypothesis of Ω to be bounded will be crucial.
- In order to proceed, it will be necessary to recall some results about measure theory and approximation of convex functions.

Moreau–Yoshida transform

Given a metric space (X, d) and a functional $F : X \to [0, +\infty]$, we define the **Moreau–Yoshida transform** by setting for every $\lambda > 0$

$$F_\lambda(x) := \inf\{F(y) + \lambda d(x, y) \ : \ y \in X\}.$$

Proposition

The functionals F_λ satisfy the following properties:

(i) for every $\lambda > 0$, the functional F_λ is λ–Lipschitz continuous.

(ii) For every $x \in X$ we have

$$\lim_{\lambda \to +\infty} F_\lambda(x) = \Gamma(X^-)F(x),$$

where $\Gamma(X^-)F$ is the *relaxed functional* of F with respect to the topology induced by d.

Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a function; we denote by f^{**} the greatest convex l.s.c. function less or equal to f. We have that

Lemma

Let $f_h : \mathbb{R}^n \to (-\infty, +\infty]$ be an increasing sequence of l.s.c. functions. Assume there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to+\infty} \frac{\theta(t)}{t} = +\infty \quad \text{and} \quad f_h(\xi) \geq \theta(|\xi|) \quad \text{for all } h \in N, \xi \in \mathbb{R}^n. \tag{2}$$

Then

$$\sup_{h\in\mathbb{N}} f_h^{**} = \left(\sup_{h\in\mathbb{N}} f_h\right)^{**}$$

In order to prove the next result we need

Lemma (continuous selections)

Let Y be a metrizable space and $T : Y \to \mathscr{P}(\mathbb{R}^k)$ be a multimapping. Assume that

i) for every $y \in Y$ the set $T(y)$ is closed, convex, and non–empty.

ii) T is l.s.c., i.e. for every open set $U \subset \mathbb{R}^k$ the following set is open:

$$T^{-}(U) := \{y \in Y : T(y) \cap U \neq \emptyset\}.$$

Then, for every $y_0 \in Y$ and every $v_0 \in T(y_0)$ there exists a continuous function $\sigma : Y \to \mathbb{R}^k$ such that

$$\begin{cases} \sigma(y_0) = v_0 \\ \sigma(y) \in T(y) \quad \text{for all } y \in Y. \end{cases}$$

Approximation of convex functions

Theorem

Let $f : \mathbb{R}^m \times \mathbb{R}^k \to (-\infty, +\infty]$ be a l.s.c. function with $f(z, \cdot)$ convex for all $z \in \mathbb{R}^m$. Assume one of the following conditions is satisfied:

i) *there exists a continuous function $\xi_0 : \mathbb{R}^m \to \mathbb{R}^k$ such that the function $z \mapsto f(z, \xi_0(z))$ is continuous and finite;*

ii) *there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = +\infty \quad \text{and} \quad f(z,\xi) \geq \theta(|\xi|) \quad \text{for every } z \in \mathbb{R}^m, \xi \in \mathbb{R}^k.$$

Then, there exists two sequences of continuous functions $a_h : \mathbb{R}^m \to \mathbb{R}^k$ and $b_h : \mathbb{R}^m \to \mathbb{R}$ such that

$$f(z,\xi) = \sup\{\langle a_h(z), \xi\rangle + b_h(z) \ : \ h \in \mathbb{N}\} \quad \text{for all } z \in \mathbb{R}^m, \xi \in \mathbb{R}^k. \tag{3}$$

Theorem

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to (-\infty, +\infty]$ *be a function which is a normal–convex integrand:*

a) $\mathscr{F} \otimes \mathscr{B}_m \otimes \mathscr{B}_k$*–measurable;*

b) $f(x, \cdot, \cdot)$ *is l.s.c. on* $\mathbb{R}^m \times \mathbb{R}^k$ *for all* $x \in \Omega \backslash N$*;*

c) $f(x, z, \cdot)$ *is convex on* $\mathbb{R}^k$ *for every* $x \in \Omega \backslash N$ *and* $z \in \mathbb{R}^m$.

If, in addition, $f(x, \cdot, \cdot)$ *for every* $x \in \Omega$ *satisfies one between conditions (i) and (ii) of the previous lemma, then there exist two sequences*

$$a_h : \Omega \times \mathbb{R}^m \to \mathbb{R}^k, \quad \textit{and} \quad b_h : \Omega \times \mathbb{R}^m \to \mathbb{R}$$

of Carathéodory integrands such that

$$f(x, z, \xi) = \sup\{\langle a_h(x, z), \xi \rangle + b_h(x, z) \ : \ h \in \mathbb{N}\} \quad \textit{for every } x \in \Omega \backslash N, z \in \mathbb{R}^m, \xi \in \mathbb{R}^k.$$

Moreover, if $f \geq 0$ *the sequences can be chosen bounded and such that*

$$f(x, z, \xi) = \sup\{[\langle a_h(x, z), \xi \rangle + b_h(x, z)]^+ \ : \ h \in \mathbb{N}\}.$$

Theorem (De Giorgi–Ioffe)

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to [0, +\infty]$ *a normal–convex integrand. Then the functional* $F : L^1_\mu(\Omega; \mathbb{R}^m) \times L^1_\mu(\Omega; \mathbb{R}^k) \to [0, +\infty]$ *defined as*

$$F(u, v) = \int_\Omega f(x, u(x), v(x))\, d\mu(x),$$

is well–posed and l.s.c. with respect to the strong topology of $L^1_\mu(\Omega; \mathbb{R}^m)$ *in the variable* u*, and the weak topology of* $L^1_\mu(\Omega; \mathbb{R}^k)$ *in the variable* v*.*

- With the aim at applying previous result to approximate f from below, we consider the additional superlinearity assumption, i.e. exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = +\infty \quad \text{and} \quad f(x, z, \xi) \geq \theta(|\xi|) \quad \text{for every } x \in \Omega, z \in \mathbb{R}^m, \xi \in \mathbb{R}^k.$$

- This can be assumed wlog since considering f as in the hypothesis of De Giorgi's Theorem and taking $u_j \to u$ in $L^1_\mu(\Omega;\mathbb{R}^m)$ and $v_j \rightharpoonup v$ in $L^1_\mu(\Omega;\mathbb{R}^k)$ we have[2] that exists a convex and superlinear $\theta : [0,+\infty) \to [0,+\infty)$ such that

$$\sup_{j\in\mathbb{N}} \int_\Omega \theta(|v_j|)d\mu \leq 1.$$

- Therefore, for every $\varepsilon > 0$ the function $f_\varepsilon(x,z,\xi) := f(x,z,\xi) + \varepsilon\theta(|\xi|)$ fulfills the superlinearity condition (beyond the others) and so, as we will show, the thesis holds for it. We conclude by noticing

$$\int_\Omega f(x,u,v)d\mu \leq \int_\Omega f_\varepsilon(x,u,v)d\mu \leq \liminf_{j\to+\infty} \int_\Omega \Big(f(x,u_j,v_j) + \varepsilon\theta(|v_j|)\Big)d\mu$$
$$= \liminf_{j\to+\infty} \int_\Omega f(x,u_j,v_j) + \varepsilon \int_\Omega \theta(|v_j|)d\mu$$

[2] Indeed $\{v_j\}_{j\in\mathbb{N}}$ is a relatively weakly compact subset in $L^1_\mu(\Omega)$ and we can apply De la Vallée–Poussin theorem [4].

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$$\begin{aligned}\int_\Omega f(x,u,v)d\mu \leq \int_\Omega f_\varepsilon(x,u,v)d\mu &\leq \liminf_{j\to+\infty} \int_\Omega \Big(f(x,u_j,v_j) + \varepsilon\theta(|v_j|)\Big)d\mu \\ &= \liminf_{j\to+\infty} \int_\Omega f(x,u_j,v_j) + \varepsilon \int_\Omega \theta(|v_j|)d\mu\end{aligned}$$

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Now, given a superlinear f that satisfies the theorem assumptions, we know that exist sequences of bounded Carathéodory integrands that approximate it:

$$f(x, u(x), v(x)) = \sup\Big\{ \Big(\langle a_h(x, u(x)), v(x) \rangle + b_h(x, u(x)) \Big)^+ \; : \; h \in \mathbb{N} \Big\}.$$

To go further we need

Lemma (De Giorgi, Buttazzo, Dal Maso)

Let $\{g_h\}_{h\in\mathbb{N}}$ be a sequence of μ–measurable functions from Ω to $(-\infty, +\infty]$ such that $g_h \geq \gamma$ for a suitable $\gamma \in L^1_\mu(\Omega)$, and $g = \sup_{h\in\mathbb{N}} g_h$. Then

$$\int_\Omega g \, d\mu = \sup_{N\in\mathbb{N}} \sup_{\{B_i^N\}} \sum_{i=1}^{N} \int_{B_i^N} g_i \, d\mu.$$

In particular for every $g \in L^1_\mu(\Omega)$ we have

$$\int_\Omega g^+ \, d\mu = \sup\Big\{ \int_B g \, d\mu \; : \; B \in \mathscr{F} \Big\}.$$

- By applying the lemma to $g_h(x)^+ := [\langle a_h(x,u(x)), v(x)\rangle + b_h(x,u(x))]^+$ we deduce the representation

$$F(u,v) = \sup_{N\in\mathbb{N}} \sup_{\{B_i^N\}} \sum_{i=1}^{N} \int_{B_i^N} \Big(\langle a_h(x,u(x)), v(x)\rangle + b_h(x,u(x))\Big)^+ d\mu(x).$$

- Summarizing: it is sufficient to show that if $a(x,z)$ and $b(x,z)$ are *bounded Carathéodory integrands* on $\Omega \times \mathbb{R}^m$, and $E \in \mathscr{F}$ then

$$G(u,v) := \int_E \Big(\langle a(x,u(x)), v(x)\rangle + b(x,u(x))\Big) d\mu(x)$$

is sequentially l.s.c. with respect to strong convergence in $L^1_\mu(E;\mathbb{R}^m)$ of u and the weak convergence in $L^1_\mu(E;\mathbb{R}^k)$ of v.

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is sequentially l.s.c. with respect to strong convergence in $L^1_\mu(E;\mathbb{R}^m)$ of u and the weak convergence in $L^1_\mu(E;\mathbb{R}^k)$ of v.

- Let us consider $u_j \to u$ in $L^1_\mu(E;\mathbb{R}^m)$ and $v_j \rightharpoonup v$ in $L^1_\mu(E;\mathbb{R}^k)$ and show that $G(u,v) \leq \liminf_j G(u_j,v_j)$.
- The latter term, not depending on v, is easier. Every subsequence $\{u_{j_s}\}_{s\in\mathbb{N}}$ has a further subsequence such that $u_{j_{s_l}} \to u$ a.e. and so

$$\int_E b(x,u(x))d\mu(x) = \lim_{l\to\infty}\int_E b(x,u_{j_{s_l}}(x))d\mu(x).$$

- We conclude that the latter holds for the whole $\{u_j\}_{j\in\mathbb{N}}$ by recalling

Proposition (Uryshon)

Let (X,d) be metric space containing x and $\{x_n\}_{n\in\mathbb{N}} \subset X$. The following are equivalent

a) The sequence $x_n \xrightarrow{d} x$.

b) Every subsequence x_{n_k} has a further subsequence $x_{n_{k_j}} \xrightarrow{d} x$.

- Let us consider $u_j \to u$ in $L^1_\mu(E;\mathbb{R}^m)$ and $v_j \rightharpoonup v$ in $L^1_\mu(E;\mathbb{R}^k)$ and show that $G(u,v) \leq \liminf_j G(u_j, v_j)$.
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Finally, we want to prove that

$$\int_E \langle a(x,u(x)), v(x)\rangle d\mu(x) = \lim_{j\to\infty} \int_E \langle a(x,u_j(x)), v_j(x)\rangle d\mu(x).$$

It is enough to ask that $g_j \to g$ *almost uniformly* on E since

Proposition

Let $\{g_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $L^\infty_\mu(E;\mathbb{R}^k)$ and $\{v_j\}_{j\in\mathbb{N}}$ a sequence in $L^1_\mu(E;\mathbb{R}^k)$ such that

- $g_j \to g$ almost uniformly on E;
- $v_j \rightharpoonup v$ in $L^1_\mu(E;\mathbb{R}^k)$.

Then

$$\int_E \langle g, v\rangle d\mu = \lim_{j\to+\infty} \int_E \langle g_j, v_j\rangle d\mu.$$

Above proposition applies to our setting. Indeed, $g_j(x) := a(x, u_j(x))$ are measurable and bounded. Moreover, by recalling the fact that $a(\cdot, \cdot)$ is Carathéodory, $u_j \to u$ in $L^1_\mu(E; \mathbb{R}^m)$, and the subsequence proposition we can conclude by using

Theorem (Severini–Egorov [5])

Let E be a measurable subset of $\mathbb{R}^n$, with $\mu(E) < +\infty$. Let $\{g_j\}_{j\in\mathbb{N}}$ be a sequence of measurable and μ–a.e. bounded functions on E. If g_j converges μ–a.e. in E to a function g which is measurable and finite, then

$$g_j \to g \quad \textit{almost–uniformly on } E.$$

Conclusions

Theorem (De Giorgi–Ioffe)

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to [0, +\infty]$ a normal–convex integrand. Then the functional $F : L^1_\mu(\Omega; \mathbb{R}^m) \times L^1_\mu(\Omega; \mathbb{R}^k) \to [0, +\infty]$ defined as

$$F(u, v) = \int_\Omega f(x, u(x), v(x))\, d\mu(x),$$

is well–posed and l.s.c. with respect to the strong topology of $L^1_\mu(\Omega; \mathbb{R}^m)$ in the variable u, and the weak topology of $L^1_\mu(\Omega; \mathbb{R}^k)$ in the variable v.

Remarks:

- The last results can be extended to non–constant sign functions, as long as the functional are well defined (see [3]).
- Provided the measure is regular enough, the convexity condition is even a necessary condition for the functional to be l.s.c.

Theorem (Olech [6])

With the further assumption that μ is a non–atomic measure, and that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^k \to [0, +\infty]$ is an integrand. If the functional

$$F(u, v) = \int_\Omega f(x, u(x), v(x))\, d\mu(x)$$

is well–defined in $L^1_\mu(\Omega; \mathbb{R}^m) \times L^1_\mu(\Omega; \mathbb{R}^k)$ with values in $[0, +\infty)$ and l.s.c. with respect to the strong topology of $L^1_\mu(\Omega; \mathbb{R}^m)$ in the variable u, and the weak topology of $L^1_\mu(\Omega; \mathbb{R}^k)$ in the variable v, then f is a normal–convex integrand.

Remarks:

- In general Olech's result does not apply to functions which depend on Du.
- If $n > 1$ for those functions the fact that "convexity implies l.s.c." is not optimal. This led to the study of weaker definitions of convexity (see [7]).

References I

- E. De Giorgi, "Teoremi di semicontinuita nel calcolo delle variazioni," *Istituto Nazionale di Alta Matematica, Roma*, vol. 1969, 1968.
- A. D. Ioffe, "On lower semicontinuity of integral functionals. i," *SIAM Journal on Control and Optimization*, vol. 15, no. 4, pp. 521–538, 1977.
- G. Buttazzo *et al.*, *Semicontinuity, relaxation and integral representation in the calculus of variations*, vol. 207.
  Longman, 1989.
- J. Alexopoulos, "de la vallée poussin's theorem and weakly compact sets in orlicz spaces," *Quaestiones Mathematicae*, vol. 17, no. 2, pp. 231–248, 1994.
- R. Beals, *Analysis: an introduction*.
  Cambridge University Press, 2004.

References II

C. Olech, "Weak lower semicontinuity of integral functionals," *Journal of Optimization Theory and Applications*, vol. 19, pp. 3–16, 1976.

C. B. Morrey Jr, "Quasi-convexity and the lower semicontinuity of multiple integrals.," *American Mathematical Society*, 1952.

Lemma (measurable selections)

Let Y be a complete separable metric space and let $T : (\Omega, \mathscr{F}) \to \mathscr{P}(Y)$ be a multimapping. Assume that

(i) fro every $x \in \Omega$ the set $T(x)$ is closed and not empty;

(ii) T is $\mathscr{F} \otimes \mathscr{B}(Y)$–measurable, in the sense that its graph $\{(x, y) \in \Omega \times Y \ : \ y \in T(x)\}$ is in $\mathscr{F} \otimes \mathscr{B}(Y)$.

Then, there exists a sequence of measurable functions $\sigma_h : (\Omega, \mathscr{F}) \to (Y, \mathscr{B}(Y))$ such that for every $x \in \Omega$ the set $\{\sigma_h(x) \ : \ h \in \mathbb{N}\}$ is dense in $T(x)$.

Theorem (De la Vallée–Poussin criterion)

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with μ positive and finite, and let $\mathscr{H} \subset L^1_\mu(\Omega)$. Then $\mathscr{H}$ is relatively compact in the weak topology of $L^1_\mu(\Omega)$ if and only if there exists a function $\theta : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{t \to +\infty} \frac{\theta(t)}{t} = +\infty, \quad \textit{and} \quad \sup\left\{ \int_\Omega \theta(|u|) d\mu \ : \ u \in \mathscr{H} \right\} < +\infty.$$